Sheet 11 Exam preparation

You will have **180 minutes** in total to complete the examination. The following rules will apply:

- Switch off your mobile phone and put it away. Do not leave it on the table.
- You may use the lecture notes of the course in paper form that may include your own annotations. No other material is admitted.
- Do not write with a pencil, nor use the colours red or green.
- All answers and solutions must provide sufficiently detailed arguments, except for multiple choice questions.
- Only one solution to each problem will be accepted. Please cross out everything that is not supposed to count.

Problem 11.1. [5 Points]

Mark all correct statements.

a) Let A, $D(A)$ be the densely defined operator

$$Af := -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}$$

with domain $D(A) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}^d) =: \mathcal{H}$. Then A is

☐ dissipative;

☐ self-adjoint;

b) The solution operator to the heat equation on $\mathbb{R}^d$, $T(t) := \mathrm{e}^{t\Delta} \in \mathrm{B}(L^2(\mathbb{R}^d))$, $t \geq 0$,

☐ is closed;

☐ is unitary;

☐ has spectrum $\sigma(T(t)) = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 11.2. [5 Points]

Let $a \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $a(x) \geq 1$ for all $x \in \mathbb{R}$ and $a, \frac{\mathrm{d}a}{\mathrm{d}x} \in L^\infty(\mathbb{R})$. Prove that for $u_0 \in H^2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \partial_x a(x) \partial_x u(t,x) + \partial_x u(t,x) \\ \quad u(0) = u_0 \end{cases}$$

admits a unique solution

$$u \in C^1([0,\infty), L^2(\mathbb{R})) \cap C^0([0,\infty), H^2(\mathbb{R})).$$

Problem 11.3. [5 Points]

We set for $f \in \mathscr{S}(\mathbb{R})$

$$\text{p.v.}\left(\frac{1}{x}\right)(f) := \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x} \mathrm{d}x + \int_{\varepsilon}^{\infty} \frac{f(x)}{x} \mathrm{d}x.$$

a) Show that $\text{p.v.}\left(\frac{1}{x}\right)$ is a well-defined tempered distribution. [3 Points]

b) Show that $\text{p.v.}\left(\frac{1}{x}\right)$ extends to a continuous linear functional on $H^2(\mathbb{R})$. [2 Points]

Problem 11.4. [5 Points]

Let $\mathcal{H}$ be a Hilbert space, A, $D(A) \subset \mathcal{H}$ be maximal dissipative and B a closed linear operator with $D(A) \subset D(B)$. Denote by $a \in [0,\infty)$ the relative A-bound of B, i.e.,

$$a := \inf\left\{\delta > 0 \,\middle|\, \exists C > 0 \text{ so that } \forall f \in D(A) : \|Bf\|_{\mathcal{H}} \leq \delta \|Af\|_{\mathcal{H}} + C\|f\|_{\mathcal{H}}\right\}.$$

Prove that

$$\limsup_{\lambda \to \infty} \|BR_\lambda(A)\|_{\mathrm{B}(\mathcal{H})} = a.$$