Sheet 9

Exercise 9.1 (Dissipative matrices)

Let $d \in \mathbb{N}$ and $A \in \mathrm{B}(\mathbb{C}^d)$ be a $d \times d$ matrix.

a) Assume there exists a unitary $U \in \mathrm{B}(\mathbb{C}^d)$ so that UAU^* is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for A to be dissipative.

b) Let $d = 2$ and A be the non-trivial Jordan block

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for A to be dissipative.

c) Let A be as in part *b)* and $\mathrm{Re}\lambda < 0$. Show that there exists a matrix S such that $B = SAS^{-1}$ is dissipative.

Exercise 9.2

Let A be maximal dissipative and $\lambda > 0$. Prove that

$$\|AR_\lambda(A)\| \leq 1.$$

Homework

Exercise 9.3 (The wave equation)

In this exercise we solve the wave equation on $\mathbb{R}^d$ using the Hille Yosida theorem. The wave equation is

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0) = u_0 \\ \partial_t u(0) = \dot{u}_0. \end{cases} \tag{W}$$

a) Let $\mathcal{H} = H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let A be the operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

with domain $D(A) = H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \tag{A}$$

then u solves the wave equation (W).

b) Show that (u, v) solves (A) if and only if $(\tilde{u}, \tilde{v}) = \mathrm{e}^{-t}(u, v)$ solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$

c) Show that $A - 1$ is maximal dissipative.

d) State the existence and uniqueness result for the wave equation implied by a)–c) and the Hille-Yosida theorem, specifying the functional space for the solution u.

Hand in on 03.04.2024 before the lecture, or electronically (the filename should start with your name) by 10:00 to `jonas.lampart@u-bourgogne.fr`.