Sheet 8

Exercise 8.1

Let $A \in \mathrm{B}(\mathbb{C}^d) = \mathbb{C}^{d\times d}$ and consider the linear autonomous ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t).$$

Show that

$$\limsup_{t\to\infty} |u(t)| < \infty$$

holds for all solutions if and only if all eigenvalues of A have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomial growth.

Exercise 8.2

Let $\mathcal{H} = L^2(\mathbb{R})$, $a \in C^1(\mathbb{R}, \mathbb{R})$ and A be the operator

$$(Af) = a'(x)f(x) + 2a(x)f'(x)$$

with domain $D(A) = C_0^\infty(\mathbb{R})$. Show that A is dissipative. Is A maximal dissipative?

Homework

Exercise 8.3 (Spectrum of multiplication operators)

Let $g : \mathbb{R} \to \mathbb{C}$ be measurable, M_g the operator of multiplication with g from Exercise 7.2 and denote by $|B|$ the Lebesgue measure of $B \in \mathscr{B}(\mathbb{R})$. Show that

a) $\sigma(M_g) = \operatorname{essran} g = \Big\{ z \in \mathbb{C} \,|\, \forall \varepsilon > 0 : \Big| \{ x \in \mathbb{R} \,|\, |z - g(x)| < \varepsilon \} \Big| > 0 \Big\}$;

b) $z \in \mathbb{C}$ is an eigenvalue of M_g if and only if

$$\big| g^{-1}(\{z\}) \big| = \big| \{ x \in \mathbb{R} : g(x) = z \} \big| > 0;$$

c) Let $g(x) := x \ \forall x \in \mathbb{R}$. Then the quantum mechanical position operator $q := M_g$ is self-adjoint, has no eigenvalues, and $\sigma(q) = \mathbb{R}$.

Hand in on 20.03.2024 before the lecture, or electronically (the filename should start with your name) by 10:00 to `jonas.lampart@u-bourgogne.fr`.