Sheet 7

Exercise 7.1

Let $A \in \mathrm{B}(\mathcal{H})$ and show that $\sigma(A)$ is compact.

Exercise 7.2

Let A, $D(A)$ be densely defined. Show that if $\rho(A) \neq \emptyset$, then A is closed.

Hint: Consider the set

$$\{(f,g) \in \mathcal{H} \times \mathcal{H} : (g,f) \in \mathscr{G}(A)\}.$$

Homework

Exercise 7.3 (Multiplication operators)

For a (possibly unbounded) measurable function $g : \mathbb{R}^d \to \mathbb{C}$ consider the linear map M_g in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_g) &:= \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, gf \in L^2(\mathbb{R}^d) \right\} \\ (M_g f)(x) &:= g(x)f(x)\,. \end{aligned}$$

Prove:

a) $\mathcal{D}(M_g)$ is dense in $L^2(\mathbb{R}^d)$.

b) $(M_g)^* = M_{\overline{g}}$.

c) M_g is closed.

d) If $g \in L^\infty(\mathbb{R}^d)$ then M_g is bounded, and

$$\|M_g\| = \|g\|_\infty = \sup \left\{ t : \left| \{ x \in \mathbb{R}^d : |g(x)| \geq t \} \right| > 0 \right\},$$

where $|V|$ denotes the Lebesgue measure of a measurable subset $V \subset \mathbb{R}^d$.

e) M_g is not bounded if $g \notin L^\infty(\mathbb{R}^d)$.

Hand in on 13.03.2024 before the lecture, or electronically (the filename should start with your name) by 10:00 to `jonas.lampart@u-bourgogne.fr`.