Sheet 6

Exercise 6.1

Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathrm{B}(\mathcal{H})$. Show that A is self-adjoint if and only if for all $f \in \mathcal{H}$

$$\langle f, Af\rangle \in \mathbb{R}.$$

Exercise 6.2

Let, for $t > 0$, $T_t \in \mathrm{B}(L^2(\mathbb{R}))$ be the solution map of the heat equation on $L^2(\mathbb{R})$,

$$(T_t f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y.$$

Show that for all $f \in \mathcal{H}$, $t > 0$ we have $\|T_t f\| < \|f\|$, and $\|T_t\| = 1$.

Homework

Exercise 6.3

Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ be bounded from below, i.e. $V(x) \geq -M$ for some $M \geq 0$ and a.e. $x \in \mathbb{R}^d$.

a) Prove that for every $f \in L^2(\mathbb{R}^d)$ and $\lambda > M$ there exists a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall \varphi \in H^1(\mathbb{R}^d) : \langle \nabla u, \nabla \varphi\rangle + \langle (V + \lambda)u, \varphi\rangle = \langle f, \varphi\rangle,$$

that is, there is a unique weak solution to the equation

$$-\Delta u(x) + V(x)u(x) + \lambda u(x) = f(x).$$

b) Prove that the weak solution $u \in H^1(\mathbb{R}^d)$ obtained in part a) is an element of $H^2(\mathbb{R}^d)$.

Hand in on 06.03.2024 before the lecture, or electronically (the filename should start with your name) by 10:00 to `jonas.lampart@u-bourgogne.fr`.