Sheet 5

Exercise 5.1

Let $\mathcal{H}$ be a Hilbert space, $f \in \mathcal{H}$ and define a linear functional by

$$\Phi(f) : \mathcal{H} \to \mathbb{C}\,, \qquad g \mapsto \langle f, g\rangle.$$

Show that

$$\|\Phi(f)\| = \|f\|.$$

Exercise 5.2

The adjoint has the following properties for $A, B \in \mathrm{B}(\mathcal{H})$ and $z \in \mathbb{C}$

a) $(A + zB)^* = A^* + \overline{z}B^*$;

b) $(AB)^* = B^*A^*$;

c) $(A^*)^* = A$

d) $\ker A^* = (\operatorname{ran} A)^\perp$ and $\ker A = (\operatorname{ran} A^*)^\perp$.

Homework

Exercise 5.3 (The Lax-Milgram Theorem)

Let $\mathcal{H}$ be a Hilbert space and

$$\alpha : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

a sesquilinear form. Asssume that

- α is *bounded*: there exists $C > 0$ so that for all $f, g \in \mathcal{H}$
$$|\alpha(f,g)| \leq C\|f\|\|g\|;$$
- α is *coercive*: there exists $a > 0$ so that for all $f \in \mathcal{H}$
$$\alpha(f,f) \geq a\|f\|^2.$$

Prove that:

a) There exists $A \in \mathrm{B}(\mathcal{H})$ so that $\alpha(f,g) = \langle Af, g\rangle$;

b) A is bijective with bounded inverse satisfying $\|A^{-1}\| \leq a^{-1}$;

c) $g = A^{-1}f$ is the unique minimiser of
$$g \mapsto \alpha(g,g) - 2\mathrm{Re}\langle f, g\rangle.$$

Hand in on 21.02.2024 before the lecture, or electronically (the filename should start with your name) by 10:00 to `jonas.lampart@u-bourgogne.fr`.