Sheet 4

Exercise 4.1

Let $k \in \mathbb{N}_0$, $f \in H^k(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$. Prove that $fg \in H^k(\mathbb{R}^d)$ and the generalised Leibniz rule holds for the derivatives of order $|\alpha| \leq k$.

Hint: You can use the convolution identity from Exercise 2.1.

Exercise 4.2

Let $\mathcal{H}$ be a Hilbert space, and $S \subset \mathcal{H}$ a subset.

a) Show that $\overline{\mathrm{span}(S)} = (S^\perp)^\perp$;

b) Deduce that
$$\overline{\mathrm{span}(S)} = \mathcal{H} \Leftrightarrow S^\perp = \{0\};$$

c) Prove that $\mathscr{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

Hint: You may use from Fourier Analysis that $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

Homework

Exercise 4.3

Prove that if X is a normed space and Y a Banach space, then $\mathrm{B}(X,Y)$ is complete, i.e., a Banach space.

Exercise 4.4 (Weak convergence)

Let $\mathcal{H}$ be a Hilbert space, $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and $f \in \mathcal{H}$. Prove that the following are equivalent as $n \to \infty$:

(i) $f_n \to f$ in norm.

(ii) $f_n \rightharpoonup f$ weakly in $\mathcal{H}$ and $\|f_n\| \to \|f\|$ as real numbers.

Hand in on 15.02.2024 before the lecture or by 10:00 by mail to `jonas.lampart@u-bourgogne.fr`.