Partial Differential Equations

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[1]These lecture notes are a draft and likely to contain mistakes. Please report any typos, errors, or suggestions to `jonas.lampart@u-bourgogne.fr`. Version of April 25, 2023

Contents

1. Introduction

A partial differential equation (PDE) is an equation whose 'unknown' is a function u, and in which (partial) derivatives of that function appear. This is similar to an ordinary differential equation (ODE) but the difference is that the unknown function

$$u : \mathbb{R}^d \to \mathbb{R} \text{ (or } \mathbb{C})$$

depends on more than one variable, $d \geq 2$, and derivatives in different directions play a role. Such equations, or systems of equations, arise in many contexts mathematics and applications in physics, engineering, and the sciences – such as electrodynamics, quantum mechanics, dynamics of weather and climate, and the description of materials.

1.1. Examples

1. The heat equation
$$\partial_t u(t,x) = \Delta_x u(t,x) \tag{1.1}$$
describes diffusion of heat in a (homogeneous, isotropic) medium.

2. Schrödinger's equation
$$\mathrm{i}\partial_t \psi(t,x) = -\Delta_x \psi(t,x) + V(x)\psi(t,x) \tag{1.2}$$
describes the wave-function of a quantum particle in an external potential V.

3. The Poisson equation
$$\Delta u(x) = \rho(x) \tag{1.3}$$
gives the electric potential generated by the (static) charge distribution ρ. Maxwell's equations give a more complete description of electrodynamics.

4. The Euler equation
$$\begin{cases} \partial_t v(t,x) + v(t,x) \cdot D_x v(t,x) + \operatorname{grad}_x p(t,x) = 0 \\ \operatorname{div}_x v(t,x) = 0 \end{cases} \tag{1.4}$$
describes the velocity field $v : \mathbb{R}^d \to \mathbb{R}^d$ and pressure $p : \mathbb{R}^d \to \mathbb{R}$ of an incompressible, inviscid fluid. Similar systems, like the Navier-Stokes equations, are used to model the dynamics of fluids and gases with different properties, e.g. water waves or atmospheric currents.

5. The Cauchy-Riemann equations

$$\begin{cases} \partial_x u(x,y) - \partial_y v(x,y) = 0 \\ \partial_y u(x,y) + \partial_x v(x,y) = 0 \end{cases} \tag{1.5}$$

are satisfied by the real and imaginary part of every holomorphic function $f = u + \mathrm{i}v : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C}$.

1.2. Linear PDEs with constant coefficients and the Fourier transform

Let $\alpha \in \mathbb{N}_0^d$ be a 'multi-index' and set

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}, \tag{1.6}$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. That is, α_j is the number of partial derivatives in direction j and $|\alpha|$ is the total number of derivatives. Since for $u \in C^k(U, \mathbb{C}^n)$ the partial derivatives can be taken in any order, we can thus express the tensor $D^k u$ by

$$(D^k u)_{j_1,\dots,j_k} = \frac{\partial^k u}{\partial x_{j_k} \cdots \partial x_{j_1}} = \partial^\alpha u \tag{1.7}$$

where α_i is the number of partial derivatives taken in the i-th direction, and $|\alpha| = k$.

Note that we have the generalised Leibniz rule

$$\partial^\alpha (fg) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g), \tag{1.8}$$

where $\beta \le \alpha$ if $\beta \le \alpha$ if $\beta_j \le \alpha_j$ for all $j = 1, \dots, d$, and the binomial coefficients are generalised as

$$\binom{\alpha}{\beta} = \prod_{j=1}^d \binom{\alpha_j}{\beta_j}. \tag{1.9}$$

Definition 1.1 (Linear PDE)**.** A PDE is called (inhomogneous) linear PDE of order k if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u = f(x), \tag{1.10}$$

where $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{n \times n}$, for $|\alpha| \le k$, and $f : \mathbb{R}^d \to \mathbb{C}^n$. The functions a_α are called the coefficients, and the PDE is called homogeneous if $f = 0$.

Question 1.2. Which of the examples in Sect. 1.1 are linear (in-) homogeneous PDEs?

A particularly simple case of linear differential equations are those with constant coefficients, where the functions $a_\alpha(x) \equiv a_\alpha$ are independent of x. These can be transformed into simpler equations by the Fourier transform. Recall that for $f \in L^1(\mathbb{R}^d)$ this is defined as

$$\hat{f}(p) = (\mathscr{F} f)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x} f(x) \mathrm{d}x. \tag{1.11}$$

Formally, we have with $p^\alpha = \prod_{j=1}^d p_j^{\alpha_j}$

$$\begin{aligned} p^\alpha \hat{f}(p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x} f(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (-\mathrm{i})^{-|\alpha|} (\partial_x^\alpha \mathrm{e}^{-\mathrm{i}p\cdot x}) f(x) \mathrm{d}x \\ &\overset{!}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} (\mathrm{i})^{|\alpha|} (-1)^{|\alpha|} \mathrm{e}^{-\mathrm{i}p\cdot x} \partial_x^\alpha f(x) \mathrm{d}x \\ &= (-\mathrm{i})^{|\alpha|} \widehat{\partial_x^\alpha f}(p), \end{aligned}$$

but the integration by parts (without boundary terms!) in the penultimate step certainly needs justification.

If we accept this identity, the linear PDE of Def. 1.1 becomes after transformation

$$\Big(\sum_{|\alpha| \le k} a_\alpha (\mathrm{i}p)^\alpha \Big) \hat{u}(p) = \hat{f}(p). \tag{1.12}$$

Any solution then satisfies, formally,

$$\hat{u}(p) \overset{!}{=} \Big(\sum_{|\alpha| \le k} a_\alpha (\mathrm{i}p)^\alpha \Big)^{-1} \hat{f}(p).$$

As the Fourier transform can be inverted, this solution should even give the unique solution. However, since the function on the right hand side may be singular, it is not clear if we can really apply the inverse Fourier transform, and in what sense this yields a solution.

1.2.1. The Fourier transform on $\mathscr{S}$

A good framework to consider identities such as (1.12) is the space of Schwartz functions, where we can

- differentiate
- multiply by polynomials
- define the Fourier transform and its inverse.

Recall (Fourier analysis):

Definition 1.3. The Schwartz space is

$$\mathscr{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha, \beta \in \mathbb{N}_0^d : \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| < \infty \right\}. \tag{1.13}$$

A sequence f_n, $n \in \mathbb{N}$ in $\mathscr{S}$ converges to $f \in \mathscr{S}$ iff

$$\forall \alpha, \beta \in \mathbb{N}_0^d : \lim_{n \to \infty} \|f_n - f\|_{\alpha,\beta} = 0, \tag{1.14}$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|. \tag{1.15}$$

A map $T : \mathscr{S}(\mathbb{R}^d) \to X$ into a metric space X is continuous iff T is sequentially continuous, that is, if for every sequence f_n converging to $f \in \mathscr{S}(\mathbb{R}^d)$

$$\lim_{n \to \infty} T f_n = T f \tag{1.16}$$

converges in X.

Question 1.4. Which of the following functions are elements of $\mathscr{S}(\mathbb{R})$?

1. $x \mapsto \cos(x)$,
2. $x \mapsto \cosh(x)^{-1} = 2(e^x + e^{-x})^{-1}$,
3. $x \mapsto e^{-|x|}$,
4. $x \mapsto e^{-|x|^2}$.

The following proposition from Fourier analysis justifies the calculations leading to (1.12).

Proposition 1.5. *The formula* (1.11) *defines a linear and continuous map*

$$\mathscr{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d), \qquad f \mapsto \hat{f}.$$

This map is invertible, with inverse

$$\mathscr{F}^{-1} f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i} p \cdot x} f(p) \mathrm{d}p.$$

Moreover, the identities

$$\partial_p^\alpha \hat{f}(p) = (-\mathrm{i})^{|\alpha|} \widehat{x^\alpha f}(p) \quad \text{and} \quad \widehat{\partial_x^\alpha f}(p) = \mathrm{i}^{|\alpha|} p^\alpha \hat{f}(p). \tag{1.17}$$

hold.

Example 1.6. Let $z \in \mathbb{C}$, $f \in \mathscr{S}(\mathbb{R}^d)$ and consider the linear PDE

$$(\Delta + z)u = f. \tag{1.18}$$

Assuming that $u \in \mathscr{S}$, we can take the Fourier transform and obtain

$$(-p^2 + z)\hat{u}(p) = \hat{f}(p). \tag{1.19}$$

If $z \in \mathbb{C} \setminus \mathbb{R}_+$, then $-p^2 + z \neq 0$, and

$$\hat{u}(p) = (-p^2 + z)^{-1}\hat{f}(p) \in \mathscr{S}. \tag{1.20}$$

In this case, the unique solution $u \in \mathscr{S}(\mathbb{R}^d)$ to (1.18) is given by

$$u(x) = \mathscr{F}^{-1}(-p^2 + z)^{-1}\hat{f}. \tag{1.21}$$

Uniqueness holds only with the requirement that $u \in \mathscr{S}$. Without this hypothesis, we can add any solution v of the homogeneous equation

$$(\Delta + z)v = 0, \tag{1.22}$$

for example $v_\pm = \mathrm{e}^{\pm\sqrt{-z}x}$ for $d = 1$, $z \neq 0$. Note that these solutions are not elements of $\mathscr{S}$, as they do not decay for $|x| \to \infty$!

If $z \in \mathbb{R}_+$ the situation is more complicated as $-p^2 + z$ is not smoothly invertible, but if $\hat{f}$ has the same zeros the solution might still be an element of $\mathscr{S}$.

Example 1.7. (The heat equation on $\mathscr{S}$) If we take the Fourier transform of the heat equation

$$\partial_t u = \Delta u \tag{1.23}$$

in both t and x, we obtain

$$(\mathrm{i}\tau + p^2)\mathscr{F}_{t,x}u = 0. \tag{1.24}$$

In the best case this would tell us that $u = 0$ (though this is not clear since the multiplier vanishes at $(\tau, p) = 0$). However, the equation is an evolution equation and $\mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ is not a natural space for the solutions. Indeed, $u \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^d)$ would mean that $u(t, x) \to 0$ for $t \to \pm\infty$, but instead of this restriction we should rather specify initial data, as for ODEs.

If we only take the Fourier transform in x, we obtain

$$\partial_t \hat{u}(t, p) = -p^2\hat{u}(t, p). \tag{1.25}$$

If we fix an initial condition $\hat{u}_0(p) = \hat{u}(0, p) \in \mathscr{S}(\mathbb{R}^d)$ the equation is an ODE initial value problem for every p. The unique solution is

$$\hat{u}(t, p) = \mathrm{e}^{-p^2 t}\hat{u}_0(p), \tag{1.26}$$

and for every $t \geq 0$ this is again an element of $\mathscr{S}(\mathbb{R}^d)$. With this we can see that there exists a unique function

$$(t, x) \mapsto u(t, x), \qquad u \in C^1((0, \infty) \times \mathbb{R}^d, \mathbb{C}), \qquad u(t, \cdot) \in \mathscr{S}(\mathbb{R}^d) \tag{1.27}$$

satisfying the heat equation (1.23) and such that

$$\lim_{t \to 0} u(t, \cdot) = u_0 \tag{1.28}$$

in $\mathscr{S}(\mathbb{R}^d)$ (Fourier analysis).

1.2.2. Tempered distributions

Our examples show that the framework of Schwartz functions is useful to prove theoretical results on PDES (existence, uniqueness), and also to obtain explicit formulas for solutions. However, it is quite restrictive, since e.g. for the heat equation we had to assume that $u_0 \in \mathscr{S}$. We can extend these considerations to a much more general framework by duality.

Definition 1.8. The space $\mathscr{S}'(\mathbb{R}^d)$ of *tempered distributions* on $\mathbb{R}^d$ is

$$\mathscr{S}'(\mathbb{R}^d) = \{\varphi : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C} \text{ linear and continuous}\}. \tag{1.29}$$

A sequence φ_n, $n \in \mathbb{N}$ is called convergent (in the distributional sense) to $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ if for all $f \in \mathscr{S}(\mathbb{R}^d)$

$$\lim_{n\to\infty} \varphi_n(f) = \varphi(f).$$

The corresponding topology is the coarsest topology such that $\varphi \mapsto \varphi(f)$ is continuous for all $f \in \mathscr{S}(\mathbb{R}^d)$.

Many functions can be identified with tempered distributions by the formula

$$\varphi_g(f) = \int_{\mathbb{R}^d} \overline{g}(x) f(x) \mathrm{d}x, \tag{1.30}$$

whenever the integral converges. In this case we will often say that $g \in \mathscr{S}'$, even though strictly speaking it is $\varphi_g \in \mathscr{S}'$, with $g \mapsto \varphi_g$ an anti-linear and injective map. However, there are also tempered distributions that are not functions, such as the Dirac distribution

$$\delta_a(f) = f(a). \tag{1.31}$$

These are sometimes written in to resemble the formula for φ_g, e.g., one writes

$$\delta_a(f) = \int \delta(x-a) f(x) \mathrm{d}x. \tag{1.32}$$

Note that this defines the "δ-function" $\delta(x-a)$.

Question 1.9. Which of the following formulas define a tempered distribution on $\mathbb{R}$?

1. $f \mapsto f'(0)$,
2. $f \mapsto \int f^2(x) \mathrm{d}x$,
3. $f \mapsto \int e^{\sqrt{1+x^2}} f(x) \mathrm{d}x$,
4. $f \mapsto \int |x| f(x) \mathrm{d}x$.

We can extend many (linear) operations on $\mathscr{S}$ to $\mathscr{S}'$ by duality, i.e. taking the transpose.

Proposition 1.10. *Let* $T : \mathscr{S}(R^d) \to \mathscr{S}(\mathbb{R}^d)$ *be linear and continuous. There exists a unique linear continuous map*

$$T' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d) \tag{1.33}$$

such that for all $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ *and* $f \in \mathscr{S}(\mathbb{R}^d)$

$$(T'\varphi)(f) = \varphi(Tf).$$

Proof. Clearly the duality formula defines a map $T' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, as $f \mapsto \varphi(Tf)$ is linear and continuous by composition. The map T' is linear since

$$(T'(a\varphi + b\psi))(f) = a\varphi(Tf) + b\psi(Tf) = a(T'\varphi)(f) + b(T'\psi)(f). \tag{1.34}$$

To keep it simple, we prove only sequential continuity of T' (the general case requires a better characterisation of the topology on $\mathscr{S}'$). Assume $\lim \varphi_n(f) = \varphi(f)$ for all $f \in \mathscr{S}(\mathbb{R}^d)$ holds for a sequence $(\varphi_n)_{n\in\mathbb{N}}$. Then

$$\lim_{n\to\infty} T'\varphi_n(f) = \lim_{n\to\infty} \varphi_n(Tf) = \varphi(Tf) = T'\varphi(f), \tag{1.35}$$

so T' is sequentially continuous. □

Examples 1.11.

a) Fourier transform $\mathscr{F}$. For $g \in \mathscr{S}(\mathbb{R}^d)$ we have

$$((\mathscr{F}^{-1})'\varphi_g)(f) = \varphi_g(\mathscr{F}^{-1}f) = \int \overline{g}(x)(\mathscr{F}^{-1}f)(x)\mathrm{d}p \overset{\text{Parseval}}{=} \int \overline{\hat{g}(p)} f(p)\mathrm{d}p = \varphi_{\hat{g}}(f), \tag{1.36}$$

so the action of $(\mathscr{F}^{-1})'$ on $\mathscr{S}'$ extends the one of $\mathscr{F}$ on $\mathscr{S}$. We will also denote this by

$$(\mathscr{F}^{-1})'\varphi = \mathscr{F}\varphi =: \hat{\varphi}. \tag{1.37}$$

b) Derivative: For any $\alpha \in \mathbb{N}^d$ we have $(\partial^\alpha)' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ linear and continuous. In this way we can define derivatives of all tempered distributions, in particular all L^2-functions.

c) Multiplication by a monomial: In this case we have $(x^\alpha)'\varphi_g = \varphi_{x^\alpha g} =: x^\alpha \varphi_g$.

d) Convolution with a Schwartz function. For fixed $g \in \mathscr{S}(\mathbb{R}^d)$, the map

$$f \mapsto g * f \tag{1.38}$$

is linear and continuous on $\mathscr{S}(\mathbb{R}^d)$. It thus extends to $\mathscr{S}'(\mathbb{R}^d)$. For suitable h, the formula

$$(g*)'\varphi_h(f) = \varphi_h(g * f) = \int \overline{h}(x) \int g(x-y)f(y)\mathrm{d}y\mathrm{d}x = \varphi_{h*Cg}(f) \tag{1.39}$$

holds with $Cg(x) := \overline{g}(-x)$. We thus define the convolution of g with $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ as

$$g *_{\mathscr{S}'} \varphi := (Cg*)'\varphi(f). \tag{1.40}$$

Definition 1.12. Let $\alpha \in \mathbb{N}^d$. The α-th *distributional derivative* on $\mathscr{S}'(\mathbb{R}^d)$ is defined as $(\partial^\alpha)_{\mathscr{S}'} := (-1)^{|\alpha|}(\partial^\alpha)'$.

Remark 1.13. The definition of $(\partial^\alpha)_{\mathscr{S}'}$ ensures that its action is compatible with the usual derivative and integration by parts: For $g \in \mathscr{S}(\mathbb{R}^d)$

$$((\partial^\alpha)_{\mathscr{S}'}\varphi_g)(f) = \int \overline{g(x)}(-1)^{|\alpha|}\partial_x^\alpha f(x)\mathrm{d}x = \int (\partial_x^\alpha \overline{g})(x) f(x)\mathrm{d}x = \varphi_{\partial^\alpha g}(f). \tag{1.41}$$

For this reason we will not distinguish $(\partial^\alpha)_{\mathscr{S}'}$ from the usual derivative by the notation. The distributional derivative is a local operation: Let $\varphi \in \mathscr{S}'$ have support in the open set $\Omega \subset \mathbb{R}^d$ (i.e.: $\operatorname{supp} f \subset \Omega^c \implies \varphi(f) = 0$), then $\operatorname{supp} \partial^\alpha \varphi \subset \Omega$.

Also note that

$$(\mathscr{F}\partial^\alpha\varphi)(f) = \varphi\left((-1)^{|\alpha|}\partial^\alpha \mathscr{F}^{-1} f\right) = \varphi\left(\mathscr{F}^{-1}(-i)^{|\alpha|}p^\alpha f\right) = \left((-i)^{|\alpha}|p^\alpha \mathscr{F}\varphi\right)(f), \tag{1.42}$$

where multiplication by p^α is defined as M'_{p^α}.

1.2.3. Elliptic PDEs and Sobolev spaces

We can now solve equations such as

$$(\Delta + z)u = f$$

even with $f \in \mathscr{S}'$ by the Fourier transform method (cf. Example 1.6). However, at first we only know that the solution u is an element of $\mathscr{S}'$. We do not, for instance, have a criterion that tells us if $u \in C^k$ and we have found a classical solution.

It is thus important to investigate further these (distributional) solutions. For a special class of constant coefficient linear PDEs, called *elliptic* this can be done quite easily and the regularity of solutions is described precisely by the *Sobolev spaces*.

Definition 1.14. Let

$$P = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \tag{1.43}$$

be a constant-coefficient differential operator of order k. The *symbol* of P is the function

$$\sigma_P(p) := \sum_{|\alpha| \le k} a_\alpha (ip)^\alpha.$$

Since

$$\mathscr{F}Pu = \sigma_P \mathscr{F}u, \tag{1.44}$$

we can solve PDEs as in Example 1.6 if σ_P is invertible for every k. However, the regularity can still be difficult to analyse. The following condition simplifies this enormously:

Definition 1.15. A constant-coefficient differential operator of order k is called *uniformly elliptic* if there exists $c > 0$ so that for all $p \in \mathbb{R}^d$

$$\sum_{|\alpha| = k} a_\alpha (ip)^\alpha \ge c|p|^k. \tag{1.45}$$

We note that this can only hold if $k = 2m$ is even, and only concerns the terms of the highest order in P. The terminology comes from the second order case, where the the condition means that the level sets of σ_P are ellipses.

We will now focus on the simplest elliptic operator $P = -\Delta$, $\sigma_P(p) = -(ip)^2 = p^2$. With some care, results for the general case can be obtained by the same arguments. Our goal is to show that if u is a solution to

$$-\Delta u = f \tag{1.46}$$

and $f \in C^m$, then $u \in C^n$ for an appropriate n (which will depend on the dimension).

Since our method relies on the Fourier transform and this is naturally defined in $\mathscr{S}$, $\mathscr{S}'$ and not C^m, we first need to study subspaces of $\mathscr{S}'$ that classify the regularity of distributions.

Definition 1.16. Let $s \in \mathbb{R}$. The *Sobolev space* of order s is the space

$$H^s(\mathbb{R}^d) := \left\{\varphi \in \mathscr{S}'(\mathbb{R}^d) : (1 + |\cdot|^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d)\right\} \tag{1.47}$$

with the norm

$$\|\varphi\|_{H^s} = \left\|(1 + |\cdot|^2)^{s/2}\hat{\varphi}\right\|_{L^2}. \tag{1.48}$$

Proposition 1.17.

a) *We have $H^s(\mathbb{R}^d) \subset H^t(\mathbb{R}^d)$ for $s \geq t$, and in particular $H^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ for all $s \geq 0$.*

b) *If $s \in \mathbb{N}$ is a non-negative integer, then $f \in H^s(\mathbb{R}^d)$ if and only if $f \in L^2(\mathbb{R}^d)$ and $\partial^\alpha f \in L^2(\mathbb{R}^d)$ for all $|\alpha| \leq s$.*

Proof. a): Let $s \geq t$. Then

$$\frac{(1 + p^2)^t}{(1 + p^2)^s} \leq C \tag{1.49}$$

for some $C > 0$. Thus for $f \in H^s$ we have $(1 + p^2)^{t/2}\hat{f} \in L^2$, because

$$\begin{aligned}\int (1 + p^2)^t |\hat{f}(p)|^2 \mathrm{d}p &= \int \frac{(1 + p^2)^t}{(1 + p^2)^s}(1 + p^2)^s |\hat{f}(p)|^2 \mathrm{d}p \\ &\leq C \int (1 + p^2)^s |\hat{f}(p)|^2 \mathrm{d}p = C\|f\|^2_{H^s}.\end{aligned} \tag{1.50}$$

Hence $f \in H^t$ and thus $H^s \subset H^t$. As $H^0 = L^2$ by definition this proves a).

b): Let first $f \in H^m(\mathbb{R}^d)$, $m \in \mathbb{N}$. Then $f \in L^2$ by a) and we have for the derivative in $\mathscr{S}'$

$$\partial^\alpha f = \mathscr{F}^{-1}(ip)^\alpha \hat{f}. \tag{1.51}$$

By Plancherel's Theorem it is thus enough to show that $(ip)^\alpha \hat{f} \in L^2$ for $|\alpha| \leq m$. This now follows from the inequalities

$$\left|(i)^{|\alpha|} p_1^{\alpha_1} \cdots p_d^{\alpha_d} \hat{f}(p)\right|^2 \leq |p|^{2|\alpha|} |f(p)|^2 \leq (1 + p^2)^{|\alpha|} |f(p)|^2 \leq (1 + p^2)^m |f(p)|^2. \tag{1.52}$$

For the reverse implication, we have by Plancherel that $(ip)^\alpha \hat{f}$ for all $|\alpha| \leq m$ and thus $p^{2\alpha}|\hat{f}(p)|^2 \in L^1$. Now

$$p^{2m} = (p_1^2 + \cdots + p_d^2)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} p^{2\alpha} \tag{1.53}$$

by the multinomial theorem, so $p^{2m}|\hat{f}(p)|^2 \in L^1$. This implies that $(1+p^2)^{m/2}\hat{f} \in L^2$ because $(1+p^{2m})/(1+p^2)^m$ is bounded, by the argument of (1.50)

□

Theorem 1.18. *For any $z \in \mathbb{C} \setminus [0,\infty)$ and $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ there exists a unique solution $u \in \mathscr{S}'(\mathbb{R}^d)$ to the equation*

$$(\Delta + z)u = \varphi.$$

Moreover, if $\varphi \in H^s(\mathbb{R}^d)$ for some $s \in \mathbb{R}$ then $u \in H^{s+2}(\mathbb{R}^d)$.

Proof. **Existence**: Since z is not a non-negative real number, $z - p^2 \neq 0$, and $(z-p^2)^{-1}$ is smooth, with bounded derivatives. Hence for $f \in \mathscr{S}(\mathbb{R}^d)$, we have $(z-p^2)^{-1} f \in \mathscr{S}(\mathbb{R}^d)$ and

$$\hat{u}(f) := \hat{\varphi}((z-p^2)^{-1} f) \tag{1.54}$$

defines an element of $\mathscr{S}'(\mathbb{R}^d)$. Setting $u = \mathscr{F}^{-1}\hat{u}$, we have for every $f \in \mathscr{S}$

$$[(\Delta + z)u](f) = u((\Delta+z)f) = \hat{u}(\mathscr{F}(\Delta+z)f) = \hat{u}((z-p^2)\hat{f}) \overset{(1.54)}{=} \hat{\varphi}(\hat{f}) = \varphi(f). \tag{1.55}$$

This means that $(\Delta + z)u = \varphi$.

Uniqueness: Let $u, v \in \mathscr{S}'$ be two, possibly different, solutions. Then for all $f \in \mathscr{S}$

$$\hat{u}((z-p^2)f) - \hat{v}((z-p^2)f) = \hat{\varphi}(\hat{f}) - \hat{\varphi}(\hat{f}) = 0. \tag{1.56}$$

Since $f \mapsto (z-p^2)\hat{f}$ is a bijection on $\mathscr{S}(\mathbb{R}^d)$ this implies that $\hat{u} = \hat{v}$, and since the Fourier transform is injective $u = v$.

Regularity: Assume that $\varphi \in H^s(\mathbb{R}^d)$, i.e. $(1+p^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d)$. First, note that $\hat{\varphi}$ is represented by a measureable function g, i.e.,

$$\hat{\varphi}(f) = \int \overline{g}(p) f(p) \mathrm{d}p. \tag{1.57}$$

Thus $\hat{u}$ is represented by the function $p \mapsto (\overline{z} - p^2)^{-1} g(p)$ and $u \in H^s(\mathbb{R}^d)$, since

$$(1+p^2)^{s/2}|\hat{u}(p)| = (1+p^2)^{s/2}\Big|\frac{g(p)}{\overline{z}-p^2}\Big| \leq C(1+p^2)^{s/2}|g(p)| \in L^2(\mathbb{R}^d). \tag{1.58}$$

Then

$$\begin{aligned}(1+p^2)^{s/2+1}\hat{u} &= (1+p^2)^{s/2}(1+p^2)\hat{u} \\ &= (1+p^2)^{s/2}(1+z)\hat{u} - (1+p^2)^{s/2}\hat{\varphi} \in L^2(\mathbb{R}^d),\end{aligned} \tag{1.59}$$

so $u \in H^{s+2}$. This proves the claim. □

Remark 1.19. We have showed that the linear map $u \mapsto (\Delta + z)u$ from $H^{s+2}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ is invertible – the inverse in the point φ is given by taking the solution to the equation above.

The link to spaces of differentiable functions is given by the following theorem.

Theorem 1.20 (Sobolev's Lemma)**.** *Let $f \in H^s(\mathbb{R}^d)$ with $s > d/2$. Then f is continuous and for all $m < s - d/2$ we have $f \in C^m(\mathbb{R}^d)$. Moreover, for $s > d/2$ and $|\alpha| \leq m < s - d/2$ there exists a constant so that for all $f \in H^s(\mathbb{R}^d)$*

$$\|\partial_x^\alpha f\|_\infty \leq C\|f\|_{H^s}.$$

For the proof, we recall from Fourier analysis:

Lemma 1.21 (Riemann-Lebesgue)**.** *If $f \in \mathscr{S}'$ such that $\hat{f} = \mathscr{F} f \in L^1(\mathbb{R}^d)$, then $f \in C(\mathbb{R}^d)$ and $\lim_{|x|\to\infty} f(x) = 0$.*

Proof of Sobolev's Lemma. We first show that $f \in H^s(\mathbb{R}^d)$, $s > d/2$ is continuous. By the Riemann-Lebesgue Lemma, it is sufficient to show that $\hat{f} \in L^1$. This follows from the Cauchy-Schwarz inequality by

$$\int |\hat{f}(p)|\mathrm{d}p = \int (1+p^2)^{-s/2}(1+p^2)^{s/2}|\hat{f}(p)|\mathrm{d}p \leq \|f\|_{H^s}\Big(\int (1+p^2)^{-s}\mathrm{d}p\Big)^{1/2}, \quad (1.60)$$

where the final integral is finite because $2s > d$. Now let $m < s - d/2$ and $|\alpha| \leq m$. Then $(ip)^\alpha \hat{f} \in L^1$, since

$$\int |(ip)^\alpha \hat{f}(p)|\mathrm{d}p \leq \int (1+p^2)^{m/2}|\hat{f}(p)| \leq \|f\|_{H^s}\Big(\int (1+p^2)^{-s+m}\mathrm{d}p\Big)^{1/2}. \quad (1.61)$$

Hence the distributional derivative $\partial^\alpha f \in \mathscr{S}'$ is a continuous function. It remains to show that this equals the usual derivative. We show this for a derivative of order one, the general case follows by repetition of the same argument. Let $\ell \in \{1, \dots, d\}$ and let $g := \mathscr{F}^{-1} ip_\ell \hat{f}$ denote the distributional derivative in direction x_ℓ. Then by the Fourier inversion formula

$$\frac{f(x+\varepsilon e_\ell) - f(x) - \varepsilon g(x)}{\varepsilon} = \frac{1}{(2\pi)^{d/2}}\int \frac{\mathrm{e}^{\mathrm{i}xp+\mathrm{i}\varepsilon p_\ell} - \mathrm{e}^{\mathrm{i}xp} - \mathrm{i}\varepsilon p_\ell \mathrm{e}^{\mathrm{i}px}}{\varepsilon}\hat{f}(p)\mathrm{d}p. \quad (1.62)$$

This converges to zero as $\varepsilon \to 0$ by the dominated convergence theorem, since by the mean-value theorem

$$\Big|\frac{\mathrm{e}^{\mathrm{i}xp+\mathrm{i}\varepsilon p_\ell} - \mathrm{e}^{\mathrm{i}xp} - \mathrm{i}\varepsilon p_\ell \mathrm{e}^{\mathrm{i}px}}{\varepsilon}\hat{f}(p)\Big| \leq 2|p_\ell||\hat{f}(p)|, \quad (1.63)$$

where the right hand side is in $L^1(\mathbb{R}^d)$ and independent of ε. This proves that $g = \partial_{x_\ell} f$, which gives the claim.

□

We can now prove our first regularity result that applies, in particular, to the solutions obtained in Theorem 1.18.

Corollary 1.22. *Let $s \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$. If $-\Delta u = f \in H^t(\mathbb{R}^d)$ for some $t \geq s-2$, then $u \in H^{t+2}(\mathbb{R}^d)$. If $m < t + 2 - d/2$ is a non-negative integer then also $u \in C^m(\mathbb{R}^d)$.*

Proof. Let $t_1 = \min\{s, t\}$. Since $u \in H^s$, $-\Delta u \in H^{t_1}$, we have

$$\underbrace{(1+p^2)^{t_1/2}\hat{u}(p)}_{\in L^2 \text{ since } t_1 \leq s} + \underbrace{p^2(1+p^2)^{t_1/2}\hat{u}(p)}_{\in L^2 \text{ since } t_1 \leq t} = (1+p^2)^{t_1/2+1}\hat{u}(p) \in L^2(\mathbb{R}^d), \tag{1.64}$$

so $u \in H^{t_1+2}(\mathbb{R}^d)$. If $t_1 = t$ (i.e., $t \leq s$) this proves the claim. Otherwise, we apply the same argument with $s' = t_1 + 2$ and conclude that $u \in H^{t_2+2}(\mathbb{R}^d)$ with $t_2 = \min\{t, s+2\}$. We repeat this until $t_n = \min\{t, s+2(n-1)\} = t$, and this proves the claim.

The second part $u \in C^m$ follows from Sobolev's Lemma. □

2. Linear operators on Hilbert spaces

In the previous chapter we were already able to solve some PDEs, but we were restricted to PDEs with constant coefficients. This restriction allowed us solve the equation using the Fourier transform, which essentially reduces the problem to the calculation of explicit integrals. Of course, in general one cannot hope to solve all PDEs in such an explicit way. For example, the stationary Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = \lambda u(x) \tag{2.1}$$

and the time-dependent Schrödinger equation

$$i\partial_t u(t,x) = -\Delta u(x) + V(x)u(x) \tag{2.2}$$

do not have constant coefficients if the potential $V(x)$ is not constant.

In this chapter we will develop general methods for treating linear equations of the form

$$Au = f \tag{2.3}$$

where $u \in X$ for an appropriate vector space X (e.g., $C^\infty(\mathbb{R}^d)$, $\mathscr{S}(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$), and A is linear.

We will introduce Hilbert spaces and their (continuous) linear transformations as a framework to study equations of this type. In the next chapter we will then use our knowledge to solve evolution equations like the time-dependent Schrödinger equation (2.2).

2.1. Banach spaces, Hilbert spaces

Recall from analysis the definition of a general norm.

Definition 2.1. Let X be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). A norm on X is a function $\|\cdot\| : X \to [0,\infty)$ such that

(i) $\|x\| = 0 \Leftrightarrow x = 0$ (definiteness)

(ii) $\|ax\| = |a|\|x\|$ for all $x \in X$ and $a \in \mathbb{C}$ (or $a \in \mathbb{R}$) (homogeneity)

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality).

The pair $(X, \|\cdot\|)$ is then called a normed space.

Question 2.2. Which of the following maps define norms?

a) $\|x\| = |x|^p$, $x \in \mathbb{C}$, $p > 0$;

b) $\|x\| = \max_{n=1,\dots d} |x_n|$, $x \in \mathbb{C}^d$;

c) $\|f\| = |f(0)|$, $f \in C(\mathbb{R}, \mathbb{C})$.

d) $\|f\|_{\alpha,\beta} := \|x^\alpha \partial^\beta f\|_\infty$, $f \in C^\infty(\mathbb{R}^d)$, $\alpha, \beta \in \mathbb{N}_0^d$ (does the answer depend on α, β ?)

A normed space $(X, \|\cdot\|)$ is a (metric) topological space. Recall the following definitions from topology.

Definition 2.3. Let $(X, \|\cdot\|)$ be a normed space.

a) The set
$$B(x, r) := \{y \in X : \|x - y\| < r\} \tag{2.4}$$
is called the open ball in X with center x and radius r.

b) A set $U \subset X$ is open if for every $y \in U$ there is $r > 0$ so that $B(y, r) \subset U$.

c) A set $A \subset X$ is called closed if $A^c = X \setminus A$ is open.

d) For $A \subset X$ the closure
$$\overline{A} := \bigcap_{\substack{A \subset B \subset X \\ B \text{ closed}}} B$$
is the smallest closed set containing A.

e) A set $A \subset X$ is called dense in X if $\overline{A} = X$.

f) A sequence $x_n \in X$, $n \in \mathbb{N}$ in X converges to $x \in X$ if
$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \|x_n - x\| < \varepsilon.$$

g) A sequence is called Cauchy sequence if
$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} : \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon.$$

h) A set $K \subset X$ is compact if every sequence $x_n \in K$, $x \in \mathbb{N}$ has a subsequence that converges to a limit $x \in K$ (this is equivalent to the definition of compactness by open covers in all metric spaces, so in particular normed spaces).

Definition 2.4. Two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a space X are called equivalent if there exist constants $C_1, C_2 > 0$ so that for all $x \in X$
$$\|x\|_1 \leq C_1 \|x\|_2$$
$$\|x\|_2 \leq C_2 \|x\|_1,$$
or equivalently
$$C_1^{-1} \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1. \tag{2.5}$$

Proposition 2.5. *Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two equivalent norms on X. Then the topologies on X induced by both norms are the same, i.e. the open sets are the same.*

Proof. We prove that if U is open for $\|\cdot\|_1$, then it is open for $\|\cdot\|_2$. This implies the statement by exchanging the roles of $\|\cdot\|_1$, $\|\cdot\|_2$.

Take $x \in U$. Then, by definition, there exists $r > 0$ so that $B_1(x, r) \subset U$ (the index indicates that the ball is with respect to the norm $\|\cdot\|_1$). Consider now $y \in B_2(x, C_1^{-1} r)$. We have

$$\|x - y\|_1 \leq C_1 \|x - y\|_2 < r, \tag{2.6}$$

so $y \in B_1(x, r)$ and thus $B_2(x, C_1^{-1} r) \subset B_1(x, r) \subset U$. We have thus found a ball for $\|\cdot\|_2$ with center x contained in U. Since $x \in U$ was arbitrary, this shown that U is open for $\|\cdot\|_2$. □

Remark 2.6. The proof of Proposition 1.17 shows that the H^m-norm, $m \in \mathbb{N}$, is equivalent to the norm

$$|||f||| := \Big(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^2}^2 \Big)^{1/2}. \tag{2.7}$$

To do analysis on such spaces, an important notion is completeness.

Definition 2.7. A normed space $(X, \|\cdot\|)$ is complete if every Cauchy sequence in X converges to some $x \in X$. A complete normed space is called a Banach space.

Example 2.8. The following spaces are Banach spaces:

a) $\mathbb{C}^d$ with *any* norm (equivalence of norms in finite dimension!).

b) $C([0, 1], \mathbb{C})$ with the maximum norm $\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$ (uniform limit theorem).

c) The space ℓ^p of p-summable sequences, $p \in [1, \infty)$

$$\ell^p = \Big\{ a_n \in \mathbb{C}, n \in \mathbb{N} : \|a\|_p := \Big(\sum_{n=1}^{\infty} |a_n|^p \Big)^{1/p} < \infty \Big\}.$$

d) $L^\infty(\mathbb{R}^d)$ with the (essential) supremum

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| := \inf\{C > 0 : |f(x)| \leq C \text{ for a.e. } x \in \mathbb{R}^d\}$$

Remark 2.9. The spaces $\mathscr{S}$ and $\mathscr{S}'$ are not Banach spaces, since their topology does not come from a norm.

Every normed space has a completion, i.e. a Banach space in which is included as a dense subset (that is, if we identify X and $\iota(X)$ in the theorem below).

Theorem 2.10. *Let $(X, \|\cdot\|)$ be a normed space. Then there exists a Banach space $(\overline{X}, |||\cdot|||)$ and an injective linear map $\iota : X \to \overline{X}$ that is an isometry, i.e. it satisfies $|||\iota(x)||| = \|x\|$ for all $x \in X$, and $\iota(X)$ is dense in $\overline{X}$.*

Proof. The space $\overline{X}$ is constructed from the space of all Cauchy sequences in X, where two sequences are identified if their difference converges to zero (much like the construction of the completion $\mathbb{R}$ of $\mathbb{Q}$). We omit the details. □

In most cases we will only consider Hilbert spaces, which additionally have a scalar product.

Definition 2.11. A (complex) Hilbert space $\mathcal{H}$ is a vector space with a scalar product (a positive definite sesquilinear form), such that $\mathcal{H}$ with the norm $\|f\| := \sqrt{\langle f, f\rangle}$ is complete (i.e. a Banach space).

Example 2.12. For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_{H^s} = \int (1 + p^2)^s \overline{\hat{f}(p)} \hat{g}(p) \mathrm{d}p.$$

Definition 2.13. Let $\mathcal{H}$ be a Hilbert space. Two vectors $f, g \in \mathcal{H}$ are called orthogonal if $\langle f, g\rangle = 0$. For a set $S \subset \mathcal{H}$, the orthogonal complement $S^\perp$ is defined by

$$S^\perp := \{f \in \mathcal{H} : \forall g \in S \, \langle f, g\rangle = 0\}.$$

Proposition 2.14. *Let $B := \overline{B(f, r)}$ be a closed ball in a Hilbert space $\mathcal{H}$. Then B is compact if and only if $\mathcal{H}$ is finite-dimensional.*

Proof. Since translation by f and scaling by r^{-1} is a homeomorphism, it is sufficient to prove the statement for $B := \overline{B(0, 1)}$.

If $\mathcal{H}$ has dimension $d < \infty$, then the unit ball is compact because (after choosing an orthonormal basis) it is a closed and bounded subset of $\mathbb{C}^d$.

Assume now that the dimension of $\mathcal{H}$ is infinite and let $f_1 \in \mathcal{H}$ be any vector with $\|f_1\| = 1$. Then $F_1 := \mathrm{span}\{f_1\}$ is a one-dimensional closed subspace of $\mathcal{H}$, and $\mathcal{H} = F \oplus F^\perp$ (cf. [FA]). Since $\dim \mathcal{H} = \infty$, $\dim (F^\perp) = \dim\mathcal{H} - 1 = \infty$, and we can choose $f_2 \in F_1^\perp$ with $\|f_2\| = 1$. Continuing in this way, we find a sequence of vectors f_n, $n \in \mathbb{N}$ satisfying $\|f_n\| = 1$ and $\langle f_n, f_m\rangle = 0$ for $n \neq m$. We thus have for all $n, m \in \mathbb{N}$

$$\|f_n - f_m\|^2 = \|f_n\|^2 + \|f_m\|^2 + 2\mathrm{Re}\langle f_n, f_m\rangle = 2, \tag{2.8}$$

so this sequence cannot contain a convergent subsequence. □

Definition 2.15. An orthonormal system (ONS) in $\mathcal{H}$ is a family $\{e_i, i \in I\} \subset \mathcal{H}$, such that

$$\langle e_i, e_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

An orthonormal system is called complete (or an orthonormal Hilbert basis) if for every $f \in \mathcal{H}$

$$f = \sum_{i\in I} \langle e_i, f\rangle e_i. \tag{2.9}$$

A Hilbert space is called separable if there exists a countable complete ONS in $\mathcal{H}$. In the following we will only consider separable Hilbert spaces.

Example 2.16. The Sobolev spaces $H^s(\mathbb{R}^d)$ are separable. A complete ONS can be given as follows. First, choose a complete ONS in $L^2(\mathbb{R}^d)$, for example the Hermite functions (here in $d = 1$)

$$e_n(x) := c_n H_n(x) e^{-\frac{1}{2}x^2}, \tag{2.10}$$

where $H_n(x)$ are the Hermite polynomials, c_n normalizing constants, and $n \in \mathbb{N}_0$. A complete ONS of $H^s(\mathbb{R}^d)$ is then given by $e_{n,s}(x) := \mathscr{F}^{-1}(1+p^2)^{-s/2}\hat{e}_n(p)$.

Remark 2.17. Note that separability does not mean that the vector-space dimension of $\mathcal{H}$ is countable (that would require the linear combination to be finite). In fact, the vector space dimension of a Hilbert space is either finite or uncountable by Baire's theorem.

Definition 2.18. Let $\mathcal{H}$ be a Hilbert space and $f_n \in \mathcal{H}$, $n \in \mathbb{N}$ be a sequence. We say that this sequence converges weakly to $f \in \mathcal{H}$ if

$$\forall g \in \mathcal{H} : \lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle.$$

Example 2.19. Let e_n, $n \in \mathbb{N}$ be an ONS in $\mathcal{H}$. Then e_n converges to zero weakly as $n \to \infty$. Indeed, for any $g \in \mathcal{H}$ we have the Bessel inequality

$$\sum_{n=1}^{\infty} |\langle g, e_n\rangle|^2 \le \|g\|^2, \tag{2.11}$$

so $\langle g, e_n\rangle$ converges to zero.

As seen above, there are bounded sequences in $H^s(\mathbb{R}^d)$ that do not have convergent subsequences. However, with the notion of weak convergence we can still find some sort of limit.

Theorem 2.20. *Let $\mathcal{H}$ be a separable Hilbert space. Any bounded sequence $f_n \in \mathcal{H}$, $n \in \mathbb{N}$, has a weakly convergent subsequence.*

Proof. Let e_n, $n \in \mathbb{N}$, be a complete ONS in $\mathcal{H}$. For every n, the sequence $k \mapsto \langle e_n, f_k\rangle$ is a bounded sequence in $\mathbb{C}$ (and thus has a convergent subsequence), since

$$|\langle e_n, f_k\rangle| \le \|f_k\| \le \sup_k \|f_k\| < \infty. \tag{2.12}$$

We will extract a joint convergent subsequence by a diagonal argument. Start with $n = 1$ by extracting a convergent subsequence, i.e., an infinite subset $S_1 \subset \mathbb{N}$ with

$$\lim_{\substack{k\to\infty \\ k\in S_1}} \langle e_1, f_k\rangle = c_1 \in \mathbb{C}. \tag{2.13}$$

The sequence $\langle e_2, f_k\rangle$, $k \in S_1$, is obviously bounded, so we can again extract a convergent subsequence $S_2 \subset S_1 \subset \mathbb{N}$. By repeating this argument, we obtain infinite sets S_j, $j \in \mathbb{N}$ with $S_j \subset S_\ell$ if $j > \ell$.

Now let k_j be the j-th element of S_j (i.e., k_1 is the smallest element of S_1, k_2 the second of S_2, etc.). Then we have

$$\lim_{j\to\infty} k_j = \infty \tag{2.14}$$

$$k_j \in \bigcap_{\ell \le j} S_\ell. \tag{2.15}$$

Consequently for all $n \in \mathbb{N}$

$$\lim_{j\to\infty} \langle e_n, f_{k_j}\rangle = c_n, \tag{2.16}$$

because $k_j \in S_n$ for $j \ge n$.

We now claim that f_{k_j} converges weakly to

$$f := \sum_{n=1}^{\infty} c_n e_n. \tag{2.17}$$

First note that, by Fatou's Lemma and Parseval's identity

$$\sum_{n=1}^{\infty} |c_n|^2 \le \operatorname{liminf}_{j\to\infty} \sum_{n=1}^{\infty} |\langle e_n, f_{k_j}\rangle|^2 = \operatorname{liminf}_{j\to\infty} \|f_{k_j}\|^2 \le \sup_k \|f_k\|^2 < \infty. \tag{2.18}$$

Hence $(c_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and $f \in \mathcal{H}$ by Parseval's identity. Now for any $g \in \mathcal{H}$ and $N \in N$

$$\begin{aligned}
|\langle g, f_{k_j} - f\rangle| &\le \sum_{n=1}^{\infty} |\langle g, e_n\rangle||\langle e_n, f_{k_j} - f\rangle| \\
&\le \sum_{n=1}^{N} |\langle g, e_n\rangle||\langle e_n, f_{k_j} - f\rangle| + \Big(\sum_{n=N+1}^{\infty} |\langle g, e_n\rangle|^2\Big)^{1/2} \|f_{k_j} - f\| \\
&\le \sum_{n=1}^{N} |\langle g, e_n\rangle||\langle e_n, f_{k_j} - f\rangle| + \Big(\sum_{n=N+1}^{\infty} |\langle g, e_n\rangle|^2\Big)^{1/2} 2\sup_k \|f_k\|.
\end{aligned} \tag{2.19}$$

Let $\varepsilon > 0$. Then, since $\langle g, e_n\rangle \in \ell^2$ and thus

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} |\langle g, e_n\rangle|^2 = \lim_{N\to\infty} \Big(\|g\|^2 - \sum_{n=1}^{N} |\langle g, e_n\rangle|^2\Big) = 0, \tag{2.20}$$

we can choose $N(\varepsilon)$ so that the second term is less than $\varepsilon/2$. The first term is then a finite sum of sequences that all converge to zero as $j \to \infty$, so we can make it smaller than $\varepsilon/2$ by choosing $j \ge j_0(N, \varepsilon)$ large enough. This proves the claim. □

2.2. Bounded linear operators

Definition 2.21. Let X, Y be normed spaces. A linear map $A : X \to Y$ is called bounded if

$$\|A\| := \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Ax\|_Y < \infty.$$

Question 2.22. Which of the following maps are bounded?

a) $T_v : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $f \mapsto f(\cdot - v)$ with $v \in \mathbb{R}^d$;

b) $\varphi_f : \mathcal{H} \to \mathbb{C}$, $g \mapsto \langle f, g \rangle$ with $f \in \mathcal{H}$;

c) $\Delta : \mathscr{S} \to \mathscr{S}$, where $\mathscr{S}$ is equipped with the L^2-norm.

Proposition 2.23. *Let X, Y be normed spaces and $A : X \to Y$ linear. The following are equivalent*

a) A is bounded;

b) A is continuous in $x = 0$;

c) A is continuous.

Proof. We prove $a) \Rightarrow b) \Rightarrow c) \Rightarrow a)$.

Let A be bounded. Since $A0 = 0$ by linearity, continuity in $x = 0$ means that

$$\forall \varepsilon > 0 \, \exists \delta > 0 : AB_X(0, \delta) \subset B_Y(0, \varepsilon). \tag{2.21}$$

We have for $x \neq 0$

$$\|Ax\|_Y = \left\| \|x\|_X A \frac{x}{\|x\|_X} \right\|_Y = \|x\|_X \left\| A \underbrace{\frac{x}{\|x\|_X}}_{\|\cdot\|_X = 1} \right\|_Y \leq \|A\| \|x\|_X, \tag{2.22}$$

so (2.21) holds with $\delta = \|A\|^{-1} \varepsilon$. This proves $a) \Rightarrow b)$.

Let A be continuous in $x = 0$. By linearity $Ax = Ax_0 + A(x - x_0)$, and by continuity in zero

$$\forall \varepsilon > 0 \, \exists \delta > 0 : \|A(x - x_0)\| < \varepsilon \text{ for } \|x - x_0\| < \delta. \tag{2.23}$$

Thus we have $AB_X(x_0, \delta) \subset B_Y(Ax_0, \varepsilon)$, which is continuity in $x = x_0$. This proves $b) \Rightarrow c)$.

To prove $c) \Rightarrow a)$ we argue by contradiction, so assume that A is continuous but not bounded. Then there exists a sequence x_n, $n \in \mathbb{N}$, with $\|x_n\| = 1$ so that

$$\lim_{n \to \infty} \|Ax_n\|_Y = \infty. \tag{2.24}$$

Consequently

$$\lim_{n \to \infty} \frac{x_n}{\|Ax_n\|_Y} = 0, \tag{2.25}$$

and by continuity of A

$$\lim_{n\to\infty} \frac{Ax_n}{\|Ax_n\|_Y} = 0. \tag{2.26}$$

But the norm of the sequence above clearly equals one, a contradiction. We thus have $c) \Rightarrow a)$. □

Theorem 2.24 (B.L.T. Theorem)**.** *Let X, Y be Banach spaces and $D \subset X$ a dense subspace. Suppose $A : D \to Y$ is a bounded linear transformation, then there exists a unique bounded linear transformation $\overline{A} : X \to Y$ that extends A, and $\|\overline{A}\| = \|A\|$ holds.*

Proof. By Proposition 2.23, A is continuous so the idea is to extend in such a way that preserves continuity.

Since $\overline{D} = X$, every $x \in X \setminus D$ is a limit point of of D (cf. Problem 8), i.e. there exist $x_n \in D$, $n \in \mathbb{N}$, so that $x_n \to x$ as $n \to \infty$. The sequence x_n is Cauchy in X, and because A is bounded, we have

$$\|Ax_n - Ax_m\|_Y \le \|A\|\|x_n - x_m\|_X, \tag{2.27}$$

so the sequence Ax_n is Cauchy in Y. Since Y is complete, it thus converges to a limit $y \in Y$. We set

$$\overline{A}x := y. \tag{2.28}$$

This is well defined, for if $\tilde{x}_n \to x$ is another sequence, then $\tilde{x}_n - x_n \to 0$ and thus

$$\lim_{n\to\infty} A\tilde{x}_n = y + \lim_{n\to\infty} A(\tilde{x}_n - x_n) = y, \tag{2.29}$$

by continuity of A. Linearity of $\overline{A}$ follows from linearity of A and the limit. This extension is unique, for if $\tilde{A}$ were another bounded extension, it would be continuous by Proposition 2.23 and $\tilde{A}x = y = \overline{A}x$ follows.

Moreover, we have by continuity of the norm

$$\|\overline{A}x\|_Y = \|\lim_{n\to\infty} Ax_n\|_Y = \lim_{n\to\infty} \|Ax_n\|_X \le \|A\| \lim_{n\to\infty} \|x_n\|_Y = \|A\|\|x\|_X, \tag{2.30}$$

so $\|\overline{A}\| \le \|A\|$. We also have $\|A\| \le \|\overline{A}\|$, since in one case the supremum is over D and in the other over X, which is larger. Thus $\overline{A}$ is bounded with $\|\overline{A}\| = \|A\|$. □

Examples 2.25.

a) The Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S} \subset L^2$ can be defined by the integral formula. We equip $\mathscr{S}$ with the L^2-norm, making it into a (incomplete) normed space, that is dense in L^2. By Plancherel, the norm of $\mathscr{F}$ is equal to one. Hence there exists a unique continuous extension $\overline{\mathscr{F}} : L^2 \to L^2$, the Fourier transform on L^2, whose norm is also one.

b) The Laplacian Δ is clearly well defined in the classical sense on $C_0^\infty(\mathbb{R}^d)$. Let $X = C_0^\infty(\mathbb{R}^d)$ with the H^2-norm (which is equivalent to the L^2-norms of the partial derivatives, see Remark 2.6). Using the Fourier transform, we see that

$$\Delta : X \to L^2(\mathbb{R}^d) \tag{2.31}$$

is bounded, because for $f \in C_0^\infty(\mathbb{R}^d)$

$$\|\Delta f\|_{L^2(\mathbb{R}^d)} = \|p^2 \hat{f}\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{H^2}. \tag{2.32}$$

There is thus a unique continuous extension to $H^2(\mathbb{R}^d)$. Since the distributional Laplacian is continuous from H^2 to L^2 and coincides with the usual Laplacian on X (since they coincide on $\mathscr{S}(\mathbb{R}^d)$) this extension is just the distributional Laplacian.

By this result, we can always assume that a bounded operator A is defined on a complete space, by passing to the completion $\overline{X}$ if necessary (cf. Theorem 2.10).

Definition 2.26. Let X, Y be normed spaces. We denote the vector space of bounded linear maps $A : X \to Y$ by $\mathrm{B}(X, Y)$. This is a normed space with

$$\|A\|_{\mathrm{B}(X,Y)} := \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Ax\|_Y.$$

- We denote by $\mathrm{B}(X)$ the space of bounded linear operators on X, $\mathrm{B}(X) := \mathrm{B}(X, X)$.
- We denote by X' the space of bounded linear functionals on X, $X' := \mathrm{B}(X, \mathbb{C})$.

Proposition 2.27. *Let X, Y be Banach spaces. Then $\mathrm{B}(X, Y)$ is complete, i.e. a Banach space.*

Proof. Exercise 11. □

On Hilbert spaces, the linear functionals have a particularly simple representation.

Theorem 2.28 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space and $\varphi \in \mathcal{H}'$ a continuous linear functional on $\mathcal{H}$. There exists a unique $f \in \mathcal{H}$ so that for all $g \in \mathcal{H}$*

$$\varphi(g) = \langle f, g \rangle.$$

The map

$$\Phi : \mathcal{H} \to \mathcal{H}', \qquad f \mapsto \langle f, \cdot \rangle$$

is an anti-linear isometric isomorphism.

Proof. Let $0 \neq \varphi \in \mathcal{H}'$. Let $K := \ker \varphi$, then K is a closed subspace and $\mathcal{H} = K \oplus K^\perp$. By the homomorphism theorem $\mathcal{H}/K \cong \operatorname{ran} \varphi = \mathbb{C}$, so $K^\perp$ has dimension one. As φ does not vanish on $K^\perp \setminus \{0\}$, there is $f \in K^\perp$ with $\varphi(f) = \|\varphi\|^2$ and we have $K^\perp = \operatorname{span}\{f\}$. For a general element of $\mathcal{H}$, $h = g + af$, $g \in K$, $a \in \mathbb{C}$, we then have

$$\varphi(g + af) \overset{g \in K}{=} \varphi(af) = a\|\varphi\|^2 = \frac{\|\varphi\|^2}{\|f\|^2} a\|f\|^2 = \frac{\|\varphi\|^2}{\|f\|^2} \langle f, g + af \rangle. \tag{2.33}$$

Moreover, we have

$$\|\varphi\| = \sup_{\substack{h\in\mathcal{H}\\ \|h\|=1}} |\varphi(h)| = \sup_{\substack{g\in K^\perp\\ \|g\|=1}} |\varphi(g)| = \varphi\Big(\frac{f}{\|f\|}\Big) = \frac{\|\varphi\|^2}{\|f\|}, \tag{2.34}$$

so $\|\varphi\| = \|f\|$ and $\varphi(h) = \langle f, h\rangle$. This shows that Φ is bijective and $\|\Phi^{-1}(\varphi)\| = \|\varphi\|$. The fact that $\|\Phi(f)\| = \|f\|$ is Exercise **??**. Clearly, Φ is anti-linear, and the proof is complete. □

This result has important consequences for tempered distributions, which are defined as linear functionals. For example, we can show that $(H^s)'$ is naturally identified with H^{-s}.

Corollary 2.29. *Let $\varphi \in \mathscr{S}'(\mathbb{R}^d)$, $s \in \mathbb{R}$ and assume that there exists a constant $C \geq 0$ so that for all $f \in \mathscr{S}(\mathbb{R}^d)$*

$$|\varphi(f)| \leq C\|f\|_{H^s}.$$

Then there exists $g \in H^{-s}(\mathbb{R}^d)$ so that

$$\varphi(f) = \int \overline{\hat{g}(p)}\hat{f}(p)\mathrm{d}p.$$

Proof. By the assumed inequality, the linear map $\varphi : \mathscr{S} \to \mathbb{C}$ is bounded with respect to the H^s-norm. As $\mathscr{S}$ is dense in H^s (see Exercise 10) it extends uniquely to a continuous linear functional on $H^s(\mathbb{R}^d)$ by the B.L.T Theorem. By the Riesz Representation Theorem, there exists $h \in H^s(\mathbb{R}^d)$ so that

$$\varphi(f) = \int (1+p^2)^s\overline{\hat{h}(p)}\hat{f}(p)\mathrm{d}p = \int \overline{\hat{g}(p)}\hat{f}(p)\mathrm{d}p \tag{2.35}$$

with $\hat{g}(p) := (1+p^2)^s\hat{h}(p)$, which is clearly an element of $H^{-s}(\mathbb{R}^d)$. □

We will now turn our focus to the operators $\mathrm{B}(\mathcal{H})$ on a separable complex Hilbert space $\mathcal{H}$.

Proposition 2.30. *Let $A \in \mathrm{B}(\mathcal{H})$. There exists a unique $A^* \in \mathrm{B}(\mathcal{H})$, called the adjoint, such that*

$$\forall f, g \in \mathcal{H} : \langle f, Ag\rangle = \langle A^* f, g\rangle.$$

Moreover, $\|A^*\| = \|A\|$.

Proof. The map

$$g \mapsto \langle f, Ag\rangle \tag{2.36}$$

is a continuous linear functional on $\mathcal{H}$. By the Riesz representation theorem, there exists $h \in \mathcal{H}$ so that

$$\langle f, Ag\rangle = \langle h, g\rangle.$$

We set $A^*f := h$ and this is well defined by uniqueness of h. The map A^* is linear, since for $f, g, h \in \mathcal{H}$, $a \in \mathbb{C}$

$$\langle A^*(f + ag), h\rangle = \langle f, Ah\rangle + \langle g, aAh\rangle = \langle A^*f + aA^*g, h\rangle. \tag{2.37}$$

Moreover, we have by the isometry of the Riesz representation

$$\|A^*\| = \sup_{\|f\|=1} \|A^*f\| = \sup_{\|f\|=1} \sup_{\|g\|=1} |\langle A^*f, g\rangle| = \sup_{\|g\|=1} \sup_{\|f\|=1} |\langle f, Ag\rangle| = \|A\|. \tag{2.38}$$

□

The following definition generalises well-known notions for matrices to $\mathrm{B}(\mathcal{H})$.

Definition 2.31. Let $A \in \mathrm{B}(\mathcal{H})$.

a) A is called *self-adjoint* if $A^* = A$;

b) A is called *unitary* if $A^*A = 1 = AA^*$;

c) A is called *normal* if $A^*A = AA^*$.

Question 2.32. Which of the following operators are normal and/or self-adjoint, unitary?

a) $M_g f = gf$ with $g \in L^\infty(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$;

b) $T_v f = f(\cdot + v)$ with $v \in \mathbb{R}^d$ on $L^2(\mathbb{R}^d)$;

c) $T_t f = f(\cdot + t)$ with $t > 0$ on $L^2(\mathbb{R}_+)$.

Example 2.33. Let $f \in L^2(\mathbb{R}^d)$ and $u(t, x)$ be the unique solution of the heat equation (cf. [FA, Thm.4.3.5])

$$\begin{cases} \partial_t u(t, x) = \Delta u(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^d \\ u(0, x) = f(x). \end{cases}$$

Then $T_t f := u(t, \cdot)$, $t \geq 0$, is self-adjoint (and hence normal) on $L^2(\mathbb{R}^d)$, but not unitary for $t > 0$. To see this, write for $t > 0$

$$u(t, x) = \int_{\mathbb{R}^d} E_t(x - y) f(y) \mathrm{d}y = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y. \tag{2.39}$$

Then, because $E(t, x - y)$ is real and symmetric under exchange of x, y, we have

$$\langle g, E_t * f\rangle = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \overline{g(x)} \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)} f(y) \mathrm{d}y \mathrm{d}x = \langle E_t * g, f\rangle, \tag{2.40}$$

so $T_t^* = T_t$. We have

$$T_t^* T_t = T_t^2 = T_{2t} \tag{2.41}$$

because $u(t + s)$ solves the heat equation with $u(t + 0) = u(t)$, and $T_{2t} \neq 1$ for $t > 0$ so T_t is not unitary. We also have $\|T_t f\| \leq \|f\|$ hence we say that T_t is a *contraction* (in fact, the inequality is strict because $\lambda = 1$ is not an eigenvalue of T_t, but the bound cannot be improved uniformly in f).

2.3. Application: An elliptic equation with variable coefficients

As an application of our results we can now study the elliptic equation

$$-\operatorname{div} M(x)\nabla u(x) + \lambda u(x) = f(x) \tag{2.42}$$

for a non-trivial coefficients matrix M. We assume that M is uniformly elliptic, that is, there exists $a > 0$ so that for all $x \in \mathbb{R}^d$, $v \in \mathbb{C}^d$

$$\langle v, M(x)v\rangle_{\mathbb{C}^d} \geq a\|v\|^2. \tag{2.43}$$

Instead of studying the equation (2.42) directly, we will first consider its weak form. Assume that $f \in L^2(\mathbb{R}^d)$, $\lambda \in \mathbb{R}$, and $u \in L^2(\mathbb{R}^d)$ solves (2.42). Then for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}\langle f, \varphi\rangle_{L^2(\mathbb{R}^d)} &= \langle -\operatorname{div} M(x)\nabla u + \lambda u, \varphi\rangle_{L^2} \\ &= \int \langle M(x)\nabla u(x), \nabla\varphi(x)\rangle_{\mathbb{C}^d} \mathrm{d}x + \lambda\langle u, \varphi\rangle_{L^2} \\ &= \langle M(x)\nabla u(x), \nabla\varphi(x)\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)} + \lambda\langle u, \varphi\rangle_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.44}$$

If M is bounded, the latter expression is well defined for $u, \varphi \in H^1(\mathbb{R}^d)$. We thus call $u \in H^1(\mathbb{R}^d)$ a weak solution to (2.42) if

$$\forall\varphi \in H^1(\mathbb{R}^d) : \langle M\nabla u, \nabla\varphi\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)} + \lambda\langle u, \varphi\rangle_{L^2(\mathbb{R}^d)} = \langle f, \varphi\rangle_{L^2(\mathbb{R}^d)}. \tag{2.45}$$

Theorem 2.34. *Let $M \in L^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ be uniformly elliptic. Then for every $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$ there exists a unique solution $u \in H^1(\mathbb{R}^d)$ to* (2.45)*.*

Proof. For $f, g \in H^1(\mathbb{R}^d)$ denote

$$\ll f, g \gg := \lambda\langle f, g\rangle_{L^2(\mathbb{R}^d)} + \langle M\nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d,\mathbb{C}^d)}. \tag{2.46}$$

This is a scalar product on $H^1(\mathbb{R}^d)$. By ellipticity of M, we have

$$\ll f, f \gg = \lambda\|f\|_{L^2}^2 + \int \underbrace{\langle M(x)\nabla f(x), \nabla f(x)\rangle}_{\geq a\|\nabla f(x)\|^2} \geq \lambda\|f\|_{L^2}^2 + a\|\nabla f\|_{L^2}^2 \geq \min\{a, \lambda\}\|f\|_{H^1}^2. \tag{2.47}$$

On the other hand, by boundedness of M,

$$\ll f, f \gg \leq \lambda\|f\|_{L^2}^2 + \|M\|_{L^\infty}\|\nabla f\|_{L^2}^2 \leq \max\{\lambda, \|M\|_\infty\}\|f\|_{H^1}^2. \tag{2.48}$$

The norm induced by $\ll f, g \gg$ is thus equivalent to the H^1-norm, so H^1 equipped with this scalar product is complete, i.e. a Hilbert space.

The right hand side of the equation satisfies

$$|\langle f, \varphi\rangle| \leq \|f\|_{L^2}\|\varphi\|_{L^2} \leq \|f\|_{L^2}\|\varphi\|_{H^1}. \tag{2.49}$$

The map $\varphi \mapsto \langle f, \varphi\rangle$ is thus a continuous linear functional on $H^1(\mathbb{R}^d)$. By the Riesz Representation Theorem there exists a unique $u \in H^1(\mathbb{R}^d)$ so that

$$\langle f, \varphi\rangle = \ll u, \varphi \gg, \tag{2.50}$$

i.e, u is the unique solution to (2.45). □

We want to establish that, when the coefficient matrix M is sufficiently regular, the weak solution obtained in this theorem is an element of $H^2(\mathbb{R}^d)$ and solves the equation (2.42) in the sense of equality in $L^2(\mathbb{R}^d)$.

To this end, we need the following Lemma on the difference quotients.

Lemma 2.35. *Define for $0 \neq h \in \mathbb{R}^d$ an operator $D_h \in \mathrm{B}(L^2(\mathbb{R}^d))$ by*

$$(D_h f)(x) = \frac{f(x+h) - f(x)}{|h|}.$$

a) If $f \in H^1(\mathbb{R}^d)$, then for all $h \in \mathbb{R}^d$: $\|D_h f\| \leq \|\nabla f\|$.

b) If $f \in L^2(\mathbb{R}^d)$ and $\sup_{0 \neq h \in \mathbb{R}^d} \|D_h f\| < \infty$, then $f \in H^1(\mathbb{R}^d)$.

Proof. a) Assume first that $f \in \mathscr{S}(\mathbb{R}^d)$. Then by the fundamental theorem of calculus

$$|D_h f(x)| = \left| \frac{1}{|h|} \int_0^1 h \cdot \nabla f(x+th) \mathrm{d}t \right| \leq \int_0^1 |\nabla f(x+th)| \mathrm{d}t. \tag{2.51}$$

Thus by Cauchy-Schwarz

$$\|D_h f\|^2 \leq \int_0^1 \int_0^1 \int_{\mathbb{R}^d} |\nabla f(x+th)||\nabla f(x+sh)| \mathrm{d}x \mathrm{d}t \mathrm{d}s \leq \|\nabla f\|^2. \tag{2.52}$$

Since $\mathscr{S}$ is dense in $H^1(\mathbb{R}^d)$, the bounded linear maps $D_h : \mathscr{S} \to L^2$ can be extended to $H^1(\mathbb{R}^d)$ with the same norm, so the inequality still holds for $f \in H^1(\mathbb{R}^d)$. This proves a).

b) Let $i \in \{1, \dots, d\}$, $n \in \mathbb{N}$ and set $h_n = n^{-1} e_i$. By hypothesis, the sequence $D_{h_n} f$ is bounded in $L^2(\mathbb{R}^d)$. Hence by Theorem 2.20 it has a weakly convergent subsequence, which we denote by the same symbols. Let $g \in L^2(\mathbb{R}^d)$ denote the weak limit and let $\varphi \in \mathscr{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Then by a change of variables and dominated convergence

$$\begin{aligned}
\langle \varphi, g \rangle &= \lim_{n \to \infty} \langle \varphi, D_{h_n} f \rangle \\
&= \lim_{n \to \infty} n \int \overline{\varphi}(x)(f(x + n^{-1} e_i) - f(x)) \mathrm{d}x \\
&= \lim_{n \to \infty} \int n(\overline{\varphi}(x - n^{-1} e_n) - \overline{\varphi}(x)) f(x) \mathrm{d}x \\
&= \int -(\partial_i \overline{\varphi}(x)) f(x) \mathrm{d}x.
\end{aligned} \tag{2.53}$$

Hence g coincides with $\partial_i f$ in $\mathscr{S}'(\mathbb{R}^d)$, whence $\partial_i f \in L^2(\mathbb{R}^d)$. In view of Proposition 1.17 this shows that $f \in H^1(\mathbb{R}^d)$. □

Theorem 2.36. *Assume the hypothesis of Theorem 2.34 and additionally that $M \in C^1(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ and that ∇M is bounded. Let u be the weak solution to* (2.45)*, then $u \in H^2(\mathbb{R}^d)$ and* (2.42) *holds in $L^2(\mathbb{R}^d)$.*

Proof. The idea is to take the derivative of the equation, but since we do not know a priori that this makes sense, we rather consider difference quotients D_h as above.

We know that $u \in H^1(\mathbb{R}^d)$, so we may take $\varphi = D_{-h}D_h u$ in (2.45). Note that we have the following identities:

$$D^*_{-h} = D_h \tag{2.54}$$
$$\nabla D_h f = D_h \nabla f \tag{2.55}$$
$$D_h(fg) = (\tau_h f)D_h g + g D_h f, \tag{2.56}$$

where $\tau_h f(x) = f(x+h)$. With this, we find from (2.45)

$$\begin{aligned}\langle f, D_{-h}D_h u\rangle &= \langle D_h M\nabla u, D_h \nabla u\rangle + \lambda\langle u, D_{-h}D_h u\rangle \\ &= \langle(\tau_h M)D_h\nabla u, D_h\nabla u\rangle + \langle(D_h M)\nabla u, D_h\nabla u\rangle + \lambda\langle u, D_{-h}D_h u\rangle. \end{aligned}\tag{2.57}$$

Using that M is elliptic, we obtain from this and Lemma 2.35

$$\begin{aligned} a\|D_h\nabla u\|^2 &\leq \langle(\tau_h M)D_h\nabla u, D_h\nabla u\rangle \\ &\overset{(2.57)}{\leq} |\langle f, D_{-h}D_h u\rangle| + |\langle(D_h M)\nabla u, D_h\nabla u\rangle| + \lambda\|u\|\|D_{-h}D_h u\| \\ &\leq (\|f\| + \lambda\|u\|)\|\nabla D_h u\| + \|D_h M\|_{L^\infty}\|\nabla u\|\|D_h\nabla u\|. \end{aligned}\tag{2.58}$$

Now we have

$$\|D_h M\|_{L^\infty} = \left\| \int_0^1 \frac{h}{|h|}\nabla M(x+th)\mathrm{d}t \right\|_{L^\infty} \leq \|\nabla M\|_{L^\infty}, \tag{2.59}$$

so dividing (2.58) by $\|\nabla D_h u\|$ yields

$$a\|D_h\nabla u\| \leq \|f\| + \lambda\|u\| + \|\nabla M\|_{L^\infty}\|\nabla u\|. \tag{2.60}$$

By Lemma 2.35 this proves that $\nabla u \in H^1(\mathbb{R}^d, \mathbb{C}^d)$, so, by Proposition 1.17, $u \in H^2(\mathbb{R}^d)$.

By Exercise 7 we thus have $M\nabla u \in H^1(\mathbb{R}^d, \mathbb{C}^d)$, and obtain from the weak form of the equation (2.45)

$$\langle f, \varphi\rangle = \langle M\nabla u, \nabla\varphi + \lambda\langle u, \varphi\rangle \tag{2.61}$$
$$= \langle -\operatorname{div} M\nabla u + \lambda u, \varphi\rangle \tag{2.62}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. Since the latter is dense in $L^2(\mathbb{R}^d)$ this implies that

$$f + \operatorname{div} M\nabla u + \lambda u \in (H^1(\mathbb{R}^d))^\perp = \{0\}, \tag{2.63}$$

that is, equation (2.42) holds. □

Remark 2.37. If the coefficients M have $k+1$ bounded derivatives and $f \in H^k(\mathbb{R}^d)$ we can iterate the reasoning of Theorem 2.36 and obtain $u \in H^{k+2}(\mathbb{R}^d)$.

2.4. Unbounded linear operators

In the previous part we considered bounded operators $A \in \mathrm{B}(\mathcal{H})$. However, this excludes differential operators, as, e.g.,

$$-\Delta : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \tag{2.64}$$

is not in $\mathrm{B}(L^2(\mathbb{R}^d))$ or $\mathrm{B}(H^2(\mathbb{R}^d))$.

In this part we will extend the theory to such linear differential operators, and general non-bounded operators.

Definition 2.38. A *densely defined operator* on a Hilbert space $\mathcal{H}$ is a pair $A, D(A)$, where

- $D(A) \subset \mathcal{H}$ is a dense subspace,
- $A : D(A) \to \mathcal{H}$ is a linear map.

Examples 2.39.

a) Let $\mathcal{H} = \ell^2(\mathbb{N})$, and

$$D(A) = c_{00}(\mathbb{N}) = \{x \in \ell^2(\mathbb{N}) : x_n \neq 0 \text{ for finitely many } n \in \mathbb{N}\}. \tag{2.65}$$

Then for *any* sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$, $(Ax)_n = a_n x_n$ is a densely defined operator.

b) Let $\mathcal{H} = L^2(\mathbb{R})$, $D(A) = C_0^0(\mathbb{R})$ and $Af = f(0)g$ for some $g \in L^2(\mathbb{R})$. This operator is densely defined as $C_0^0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ [FA, Thm.3.4.2].

c) Let $\mathcal{H} = L^2(\mathbb{R})$, $D(A) = H^1(\mathbb{R})$, then $(Af)(x) := a(x)f'(x)$ is a densely defined operator for every $a \in L^\infty(\mathbb{R})$.

Definition 2.40. A densely defined operator $B, D(B)$ extends $A, D(A)$, if $D(A) \subset D(B)$ and $Bf = Af$ for all $f \in D(A)$. We write $A \subset B$.

If $A, D(A)$ is bounded in the sense of Definition 2.21, then by the B.L.T. theorem there exists a unique bounded extension $\overline{A}$, $D(\overline{A}) = \mathcal{H}$. In this case we usually identify A and its canonical extension.

For a linear map $A : D(A) \to \mathcal{H}$ (with $D(A) \subset \mathcal{H}$ not necessarily dense), the graph is given by

$$\mathscr{G}(A) := \{(f, Af) : f \in D(A)\} \subset D(A) \times \mathcal{H} \subset \mathcal{H} \times \mathcal{H}. \tag{2.66}$$

Since A is linear, $\mathscr{G}(A)$ is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$.

Definition 2.41. The operator $A, D(A)$ is called *closed* if the set $\mathscr{G}(A)$ is closed in $\mathcal{H} \times \mathcal{H}$, i.e., for any sequence $(f_n)_{n\in\mathbb{N}}$ in $D(A)$ such that f_n converges to $f \in \mathcal{H}$ and Af_n converges to $g \in \mathcal{H}$, it holds that $f \in D(A)$ and $Af = g$.

The operator A, $D(A)$ is called *closable* if it has a closed extension.

Proposition 2.42. *A densely defined operator $A, D(A)$ on $\mathcal{H}$ is closable if and only if $\overline{\mathscr{G}(A)}$ is the graph of an operator $\overline{A}, D(\overline{A})$. For every closed extension B of A we have $\overline{A} \subset B$.*

Proof. Assume first that A is closable, so there exists a closed extension $B \supset A$. We have

$$\mathscr{G}(A) \subset \overline{\mathscr{G}(A)} \subset \mathscr{G}(B). \tag{2.67}$$

Since $\mathscr{G}(B)$ is the graph of a linear map, it has the property that if $(f, g) \in \mathscr{G}(B)$, then $(f, h) \notin \mathscr{G}(B)$ for $g \neq h$ (i.e., $Bf = g$ is well defined). This property then also holds for any subset. We may thus define

$$\begin{aligned} D(\overline{A}) :=& \{f \in \mathcal{H} : \exists g_f \in \mathcal{H} \text{ with } (f, g_f) \in \overline{\mathscr{G}(A)}\}, \\ \overline{A}f =& g_f. \end{aligned} \tag{2.68}$$

Clearly $\overline{A} \subset B$ is an operator and $\mathscr{G}(\overline{A}) = \overline{\mathscr{G}(A)}$ is closed, proving the first implication.

The converse is obvious, for if $\overline{\mathscr{G}(A)}$ is the graph of an operator, then A has a closed extension. □

The operator $\overline{A}$ is the minimal closed extension of A and thus called the closure of A. Note that in general we do *not* have $D(\overline{A}) = \overline{D(A)}$ (only for bounded A).

Proposition 2.43. *Let $A, D(A)$ be densely defined and define the graph norm on $D(A)$ by*

$$\|f\|_{D(A)} = \sqrt{\|f\|_{\mathcal{H}}^2 + \|Af\|_{\mathcal{H}}^2} = \|(f, Af)\|_{\mathcal{H}\oplus\mathcal{H}}. \tag{2.69}$$

Then $A : D(A) \to \mathcal{H}$ is continuous w.r.t. this norm. Moreover, A is closed if and only if $(D(A), \|\cdot\|_{D(A)})$ is complete.

Proof. It clearly holds that

$$\|Af\|_{\mathcal{H}} \leq \sqrt{\|f\|_{\mathcal{H}}^2 + \|Af\|_{\mathcal{H}}^2} = \|f\|_{D(A)}, \tag{2.70}$$

so $A : D(A) \to \mathcal{H}$ is bounded w.r.t. the graph norm and thus continuous.

Assume now that A is closed and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(D(A), \|\cdot\|_{D(A)})$. Then, since $\|f\|_{\mathcal{H}} \leq \|f\|_{D(A)}$ (i.e., the embedding of $D(A)$ in $\mathcal{H}$ is continuous), $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{H}$. By the same reasoning, $(Af_n)_{n\in\mathbb{N}}$ is also Cauchy in $\mathcal{H}$. Since $\mathcal{H}$ is complete, we have $f_n \to f$ and $Af_n \to g$ for some $f, g \in \mathcal{H}$. Closedness of A implies that $f \in D(A)$ and $Af = g$. We then have

$$\lim_{n\to\infty} (\|f_n - f\|_{\mathcal{H}}^2 + \|Af_n - Af\|_{\mathcal{H}}) = 0, \tag{2.71}$$

so $(f_n) \to (f)$ in $D(A)$ and $D(A)$ is complete.

The converse is immediate from the definitions. □

Examples 2.44.

a) The operator of Example 2.39a) is always closable, with

$$D(\overline{A}) = \{x \in \ell^2 : (a_n x_n)_{n\in\mathbb{N}} \in \ell^2\}, \tag{2.72}$$

since the graph norm is just the weighted ℓ^2-norm $\left(\sum_n (1+|a_n|^2)x_n^2\right)^{1/2}$, which is complete.

b) The operator of 2.39b) is not closable. We have $\overline{\mathscr{G}(A)} = L^2(\mathbb{R}) \times \operatorname{span}\{g\}$ (which is not a graph!), since for any $f \in \mathcal{H}$, $z \in \mathbb{C}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in C_0^0(\mathbb{R})$ with $f(0) = z$ and $f_n \to f$ in L^2.

We want to define the adjoint of a densely defined operator A, $D(A)$ as in the bounded case. It should satisfy the formula

$$\langle Af, g\rangle = \langle f, A^* g\rangle. \tag{2.73}$$

The question is for what vectors f, g. For $A \in \mathrm{B}(\mathcal{H})$, we could take any $f, g \in \mathcal{H}$ and this formula defined $A^* \in \mathrm{B}(\mathcal{H})$. For unbounded A we certainly want $f \in D(A)$, but we also need to decide what $D(A^*)$ should be. The following definition chooses $D(A^*)$ in a maximal way so that the formula holds.

Definition 2.45. Let $A, D(A)$ be densely defined on $\mathcal{H}$. We define the *adjoint* A^*, $D(A^*)$ by

$$\begin{aligned} D(A^*) &:= \{g \in \mathcal{H} : \exists h_g \in \mathcal{H}\, \forall f \in D(A) : \langle Af, g\rangle = \langle f, h_g\rangle\}, \\ A^* &: D(A^*) \to \mathcal{H}, \\ A^* g &:= h_g \end{aligned}$$

Remarks 2.46.

- $A^* g$ is well defined, since if h_g exists it is unique, by

$$\forall f \in D(A) : \langle f, h_g - \tilde{h}_g\rangle = 0 \implies h_g = \tilde{h}_g, \tag{2.74}$$

  because $D(A)$ is dense.

- The requirement on $D(A^*)$ can be read as: $g \in D(A^*) \Leftrightarrow$ the linear functional $f \mapsto \langle g, Af\rangle$ on $D(A)$ extends continuously to $\mathcal{H}$, since then h exists by the Riesz Representation Theorem.

- $A \subset B \implies B^* \subset A^*$, since there are fewer conditions to be met in $D(A^*)$, and for $f \in D(B^*) \subset D(A^*)$, $g \in D(A) \subset D(B)$

$$\langle B^* f, g\rangle = \langle f, Bg\rangle \overset{A\subseteq B}{=} \langle f, Ag\rangle = \langle A^* f, g\rangle. \tag{2.75}$$

- $D(A^*)$ is *not* always dense.

- *If* $D(A^*)$ is dense we can define $A^{**} = (A^*)^*$.

Proposition 2.47. *Let A, $D(A)$ be densely defined on $\mathcal{H}$.*

a) $\mathscr{G}(A^*)$ is closed.

b) $D(A^*)$ is dense if and only if A is closable.

c) If A is closable then $\overline{A} = A^{**}$ and $A^* = (\overline{A})^*$.

Proof. a) Let (g_n, h_{g_n}) be a sequence in $\mathscr{G}(A^*)$ that converges to $(g, h) \in \mathcal{H} \times \mathcal{H}$. Then for all $f \in D(A)$:

$$\underbrace{\langle Af, g_n \rangle}_{\to \langle Af, g \rangle} = \underbrace{\langle f, h_{g_n} \rangle}_{\to \langle f, h \rangle},$$

so $g \in D(A^*)$ and $A^* g = h$, whence $(g, h) \in \mathscr{G}(A^*)$.

b) If A^* is densely defined, then A^{**} extends A, because for every $g \in D(A)$ there exists $h = Ag \in \mathcal{H}$ such that

$$\forall f \in D(A^*) : \langle A^* f, g \rangle = \langle f, h \rangle. \tag{2.76}$$

By a), A^{**} is closed and thus A is closable.

Assume now that A^* is not densely defined and consider $\overline{\mathscr{G}(A)} = (\mathscr{G}(A)^\perp)^\perp$ (compare Exercise 10). Note that

$$\mathscr{G}(A^*) = \{(g, h) \in \mathcal{H} \times \mathcal{H} : \forall f \in D(A) : \langle Af, g \rangle - \langle f, h \rangle = 0\}, \tag{2.77}$$

and since $\langle Af, g \rangle - \langle f, h \rangle = \langle (f, Af), (-h, g) \rangle_{\mathcal{H} \oplus \mathcal{H}}$, we have

$$\mathscr{G}(A)^\perp = \{(-A^* g, g) : g \in D(A^*)\}. \tag{2.78}$$

Now let $0 \neq \xi \in D(A^*)^\perp$, and observe that $(0, \xi) \in (\mathscr{G}(A)^\perp)^\perp$, but certainly not in the graph of any linear operator.

c) We have by (2.77),(2.78)

$$\mathscr{G}(A^{**}) = \{(g, h) \in \mathcal{H} \times \mathcal{H} : \forall f \in D(A^*) : \langle A^* f, g \rangle - \langle f, h \rangle = 0\} = (\mathscr{G}(A)^\perp)^\perp, \tag{2.79}$$

so $\overline{A} = A^{**}$. This, together with a), implies

$$\overline{A}^* = A^{***} = \overline{A^*} \overset{a)}{=} A^*. \tag{2.80}$$

□

Definition 2.48. We call a densely defined operator

- *symmetric* if $A \subset A^*$, that is,

$$\forall f, g \in D(A) : \langle Af, g \rangle = \langle f, Ag \rangle. \tag{2.81}$$

- *self-adjoint* if $A = A^*$, that is A is symmetric and $D(A^*) = D(A)$.

Theorem 2.49. *Let A, $D(A)$ be symmetric. The following are equivalent*

1) $A = A^*$

2) A *is closed and* $\ker(A^* + \mathrm{i}) = \ker(A^* - \mathrm{i}) = \{0\}$

3) $\operatorname{ran}(A + \mathrm{i}) = \operatorname{ran}(A - \mathrm{i}) = \mathcal{H}$.

Proof. 1) $\implies$ 2): If A is self-adjoint, then $A = A^*$ is closed by Theorem 2.47. If moreover $f \in \ker(A \pm \mathrm{i})$, then (e.g. for "$-$")

$$\mathrm{i}\langle f, f\rangle = \langle f, Af\rangle \overset{A=A^*}{=} \langle Af, f\rangle = -\mathrm{i}\langle f, f\rangle, \tag{2.82}$$

so $f = 0$.

2) $\implies$ 3) First note that $\ker(A^* + \mathrm{i}) = \operatorname{ran}(A - \mathrm{i})^\perp$, since

$$f \in \ker(A^* + \mathrm{i}) \Leftrightarrow \forall g \in D(A) : \langle (A^* + \mathrm{i})f, g\rangle = 0 \Leftrightarrow \forall g \in D(A) \langle f, (A - \mathrm{i})g\rangle = 0.$$

Consequently, $\ker(A^* + \mathrm{i}) = \{0\} \implies \overline{\operatorname{ran}(A - \mathrm{i})} = \mathcal{H}$ and the range of $A \pm \mathrm{i}$ is at least dense. Now let $g \in \mathcal{H}$ and choose f_n so that $(A - \mathrm{i})f_n \to g$. For all $f \in D(A)$ we have the inequality

$$\|(A - \mathrm{i})f\|^2 = \langle (A - \mathrm{i})f, (A - \mathrm{i})f\rangle = \|Af\|^2 + \|f\|^2 + 2\mathrm{Re}\langle \mathrm{i}f, Af\rangle \geq \|f\|^2 . \tag{2.83}$$

Thus the sequence f_n is Cauchy and converges to some $f \in \mathcal{H}$. Since A is closed, $f \in D(A)$ and $Af = g + \mathrm{i}f$, whence $g \in \operatorname{ran}(A - \mathrm{i})$.

3) $\implies$ 1) To show that $A = A^*$ we need to prove that $A^* \subset A$, since $A \subset A^*$ is assumed. Let $f \in D(A^*)$. Since $A - \mathrm{i}$ is onto, there is $g \in D(A)$ s.th. $(A^* - \mathrm{i})f = (A - \mathrm{i})g$. Since $A \subset A^*$, we thus have $(A^* - \mathrm{i})(f - g) = 0$. Then for every $h \in D(A)$:

$$0 = \langle h, (A^* - \mathrm{i})(f - g)\rangle = \langle (A + \mathrm{i})h, (f - g)\rangle, \tag{2.84}$$

and thus $f = g \in D(A)$ because $\operatorname{ran}(A + \mathrm{i}) = \mathcal{H}$. Thus $D(A^*) \subset D(A)$ and $A = A^*$ since $Af = A^*f$ for $f \in D(A)$. □

Examples 2.50.

1) The Laplacian Δ is self-adjoint on $D(\Delta) = H^2(\mathbb{R}^d)$. To see this, we can use the criterion 3) above. Let $f \in L^2(\mathbb{R}^d)$. Then

$$g := \mathscr{F}^{-1}(-p^2 \pm \mathrm{i})^{-1}\hat{f}(p) \in H^2(\mathbb{R}^d) \tag{2.85}$$

and

$$(\Delta \pm \mathrm{i})g = f. \tag{2.86}$$

2) The operator defined by $(Ax)_n := a_n x_n$ in 2.39 is not self-adjoint, since it is not closed. Its closure (see Example 2.44) is self-adjoint if $(a_n)_{n\in\mathbb{N}}$ is real (compare Problem **??**).

3) The operator $A = -\Delta$ with domain $D(A) = C_0^\infty(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ is not self-adjoint, nor is its closure. To see this, let $f(x) = \mathrm{e}^{-\sqrt{\mathrm{i}}x}$ (where $\mathrm{Re}\sqrt{\mathrm{i}} > 0$) and observe that for all $g \in C_0^\infty$ we can integrate by parts without boundary terms, so

$$-\langle f, \Delta g\rangle = -\int_0^\infty \bar{f}(x)\Delta g(x)\mathrm{d}x = \int_0^\infty (-\Delta \bar{f}(x))g(x)\mathrm{d}x = \langle(-\mathrm{i}f), \Delta g\rangle. \tag{2.87}$$

Thus $f \in D(A^*)$ and $(A^* + \mathrm{i})f = 0$, and also $f \in \ker(\bar{A}^* + \mathrm{i})$ by Proposition 2.47.

2.5. Spectrum and resolvent

In this section A, $D(A)$ is a densely defined operator on the *complex* Hilbert space $\mathcal{H}$. This includes the bounded case $A \in \mathrm{B}(\mathcal{H})$ where $D(A) = \mathcal{H}$.

Definition 2.51. Let $A, D(A)$ be densely defined on $\mathcal{H}$. The set

$$\rho(A) := \{z \in \mathbb{C} : A - z : D(A) \to \mathcal{H} \text{ is bijective, and } (A-z)^{-1} \text{ is bounded}\} \tag{2.88}$$

is called the *resolvent set* of A. For $z \in \rho(A)$ the operator

$$R_z(A) := (A - z)^{-1} \tag{2.89}$$

is called the *resolvent*.

Definition 2.52. The complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the *spectrum* of A. We define

- The *point spectrum*

$$\sigma_p(A) := \{z \in \mathbb{C} : A - z \text{ is not one-to-one}\}$$

- The *continuous spectrum*

$$\sigma_c(A) := \{z \in \mathbb{C} : A - z \text{ is one-to-one, } \mathrm{ran}(A-z) \neq \mathcal{H} \text{ but } \overline{\mathrm{ran}(A-z)} = \mathcal{H}\}$$

- The *residual spectrum*

$$\sigma_r(A) := \{z \in \mathbb{C} : A - z \text{ is one-to-one but } \overline{\mathrm{ran}(A-z)} \neq \mathcal{H}\}.$$

Examples 2.53.

1) If $\dim \mathcal{H} < \infty$ then the spectrum coincides with the set of eigenvalues $\sigma(A) = \sigma_p(A)$, since if $z \in \mathbb{C}$ is injective then

$$\dim \mathrm{ran}(A - z) = \dim \mathcal{H} - \dim \ker(A - z) = \dim \mathcal{H}, \tag{2.90}$$

so $A - z$ is bijective. Note that $\sigma(A)$ is not empty since we assume $\mathcal{H}$ to be a complex vector space.

2) Take $\mathcal{H} = \ell^2$ and $(Ax)_n = a_n x_n$ as in Example 2.39a). Then $\sigma(A) = \mathbb{C}$, since $\operatorname{ran}(A - z) \subset c_{00} \neq \mathcal{H}$. However $\sigma(\overline{A}) = \overline{\sigma_p(\overline{A})} = \overline{\cup_n \{a_n\}}$, since for z not an accumulation point of $(a_n)_{\mathbb{N}}$ the formula

$$\left(R_z(\overline{A})x\right)_n = (a_n - z)^{-1} x_n \tag{2.91}$$

defines the resolvent. Thus $\sigma(A)$ depends strongly on $D(A)$!

Proposition 2.54. *Let A, $D(A)$ be closed. Then*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A). \tag{2.92}$$

Proof. One needs to prove that if $A - z : D(A) \to \mathcal{H}$ is bijective, then the inverse $(A - z)^{-1}$ is continuous. Using that $D(A)$ is a Banach space, this follows from a fundamental theorem in functional analysis, the open mapping theorem, that we will not prove here, see [Bre, Cor.2.7]. □

Lemma 2.55 (Neumann series)**.** *Let $A \in \mathrm{B}(\mathcal{H})$ with $\|A\| < 1$. Then $1 - A$ is bijective and*

$$(1 - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Proof. We show first that the Neumann series converges, and then that it is the inverse of $1 - A$. Since $\|A^k\| \leq \|A\|^k$, we have

$$\Big\| \sum_{k=n}^{m} A^k \Big\| \leq \sum_{k=n}^{\infty} \|A\|^k = \frac{\|A\|^n}{1 - \|A\|}. \tag{2.93}$$

Hence the sequence of partial sums is Cauchy and by completeness of $\mathrm{B}(\mathcal{H})$ (Proposition 2.27) it converges to some operator $B \in \mathcal{H}$. Then

$$AB = \sum_{k=1}^{\infty} A^k = B - 1 = BA, \tag{2.94}$$

so

$$(1 - A)B = 1 = B(1 - A), \tag{2.95}$$

and thus $B = (1 - A)^{-1}$. □

Theorem 2.56. *Let $A, D(A)$ be densely defined on $\mathcal{H}$. The resolvent set $\rho(A)$ is open, and $R_z(A)$ defines an analytic function $\rho(A) \to \mathrm{B}(\mathcal{H})$. Moreover, for $z, w \in \rho(A)$ the first resolvent formula*

$$R_z(A) - R_w(A) = (z - w) R_z(A) R_w(A) \tag{2.96}$$

holds, and, in particular, $R_z(A)$ and $R_w(A)$ commute.

Proof. If $\rho(A) = \emptyset$ (which is possible!) there is nothing to prove, so assume there is $z_0 \in \rho(A)$. We have

$$R_{z_0}(A)(A - z) = 1 - R_{z_0}(A)(z - z_0) = (A - z)R_{z_0}(A). \tag{2.97}$$

For $|z - z_0| < \|R_{z_0}(A)\|^{-1}$, the operator on the right hand side is invertible by a Neumann series,

$$\Big(1 - R_{z_0}(A)(z - z_0)\Big)^{-1} = \sum_{k=0}^{\infty} R_{z_0}(A)^k (z - z_0)^k. \tag{2.98}$$

Thus

$$\Big(1 - R_{z_0}(A)(z - z_0)\Big)^{-1} R_{z_0}(A)(A - z) = 1 \tag{2.99}$$

and we have a left inverse for $z - A$. The same argument also provides a right inverse, and thus

$$B(z_0, \|R_{z_0}(A)\|^{-1}) \subset \rho(A) \tag{2.100}$$

and $\rho(A)$ is open. Moreover, $R_z(A)$ is given by a convergent power series in $z - z_0$, so $z \mapsto R_z(A)$ is analytic.

The resolvent formula follows from the simple calculation

$$R_z(A) - R_w(A) = R_z(A)(A - w)R_w(A) - R_z(A)(A - z)R_w(A) = (z - w)R_z(A)R_w(A). \tag{2.101}$$

□

Proposition 2.57. *Let A, $D(A)$ be a symmetric operator on the complex Hilbert space $\mathcal{H}$. Then the spectrum of A is one of the following*

1) The complex plane;

2) The closed upper half plane;

3) The closed lower half plane;

4) A subset of the real line.

Proof. For symmetric A, $f \in D(A)$ and $z = \lambda + \mathrm{i}\mu \in \mathbb{C}$ we have

$$\|(z - A)f\|^2 = \|(\lambda - A)f\|^2 + \mu^2\|f\|^2 - 2\mathrm{Re}\langle \mathrm{i}\mu f, Af\rangle \geq \mu^2\|f\|^2. \tag{2.102}$$

Now assume that $\rho(A) \neq \emptyset$, i.e. we are not in the case 1). If $z \in \mathbb{R} \cap \rho(A)$, then by Theorem 2.56 there are points in $\rho(A)$ in both the upper and lower half planes. We can conclude the proof by showing that if $\rho(A)$ contains a point in the upper (lower) half plane, then the whole open half plane is contained in the resolvent set. To see this, let $z \in \rho(A)$ with $\mathrm{Im} z \neq 0$. Then taking (2.102) with $f = R_{z_0}(A)g$, $g \in \mathcal{H}$, gives

$$\|R_z(A)g\| \leq |\mathrm{Im} z|^{-1}\|g\|, \tag{2.103}$$

so

$$\|R_z(A)\| \le |\mathrm{Im} z|^{-1}. \tag{2.104}$$

By (2.100) the resolvent set then contains the ball $B(z, |\mathrm{Im} z|)$ which lies in the same half plane as z and touches the real line. Applying the same reasoning to further points in this ball (e.g., $\lambda + 3/2\mathrm{i}\mu$, $\lambda \pm \mu/2 + \mathrm{i}\mu$), we can enlarge the known resolevent set until it covers the whole half plane. Hence the resolvent set is either empty, an open half plane, or contains both open half planes, which corresponds to the statements on the spectrum. □

Corollary 2.58. *If A, $D(A)$ is symmetric and there exists $\lambda \in \mathbb{R} \cap \rho(A)$ then A is self-adjoint.*

Proof. If $\lambda \in \mathbb{R} \cap \rho(A)$ then by Proposition 2.57 the resolvent set of A contains the upper and lower half planes, i.e. we are in case 4) above. But then $\pm\mathrm{i} \in \rho(A)$, so $\mathrm{ran}(A \pm \mathrm{i}) = \mathcal{H}$ and thus A is self-adjoint by Theorem 2.49. □

Example 2.59. The Corollary shows that the operator $A = \mathrm{div}\, M\nabla$ with domain $D(A) = H^2(\mathbb{R}^d)$ is self-adjoint under the hypothesis of Theorem 2.36 (M elliptic with one bounded derivative). First, A is symmetric if for all $x \in \mathbb{R}^d$ the matrix $M(x)$ is self-adjoint, since then for $f, g \in H^2(\mathbb{R}^d)$

$$\langle f, Ag\rangle = -\int \langle \nabla f(x), M(x)\nabla g(x)\rangle_{\mathbb{C}^d} \mathrm{d}x = -\int \langle M(x)\nabla f(x), \nabla g(x)\rangle_{\mathbb{C}^d} \mathrm{d}x = \langle Af, g\rangle. \tag{2.105}$$

Moreover, Theorem 2.36 states that for all $f \in L^2(\mathbb{R}^d)$, $\lambda > 0$ there exists a unique solution of

$$(\lambda - \mathrm{div}\, M\nabla)u = f, \tag{2.106}$$

and $u \in H^2(\mathbb{R}^d) = D(A)$, so $(0, \infty) \in \rho(A)$.

3. Linear evolution equations

In this chapter, we will study the "initial value problem", also called the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Au \\ u(0) = u_0 \end{cases} \tag{3.1}$$

for suitable densely-defined operator A, $D(A)$ on $\mathcal{H}$.

3.1. The exponential of a bounded operator

The simplest case for (3.1) is when $A \in \mathrm{B}(\mathcal{H})$ is bounded (and thus $D(A) = \mathcal{H}$). This case is very similar to linear ODEs.

Lemma 3.1. *Let $A \in \mathrm{B}(\mathcal{H})$. Then the exponential series*

$$\mathrm{e}^A := \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

converges in $\mathrm{B}(\mathcal{H})$*, and*

$$\|\mathrm{e}^A\| \leq \mathrm{e}^{\|A\|}.$$

Proof. We have

$$\|A^j\| \leq \|A\|^j, \tag{3.2}$$

and thus

$$\Big\| \sum_{j=n}^{m} \frac{A^j}{j!} \Big\| \leq \sum_{j=n}^{m} \frac{\|A\|^j}{j!} \leq \mathrm{e}^{\|A\|} - \sum_{j=0}^{n-1} \frac{\|A\|^j}{j!}. \tag{3.3}$$

The right hand side converges to zero for $n \to \infty$ since the exponential series of real numbers converges. The sequence of partial sums is thus Cauchy in $\mathrm{B}(\mathcal{H})$ and by completeness it has a limit e^A. □

Theorem 3.2. *Let $A \in \mathrm{B}(\mathcal{H})$. For every $u_0 \in \mathcal{H}$,*

$$u = \mathrm{e}^{tA} u_0 \in C^\infty(\mathbb{R}, \mathcal{H}) \tag{3.4}$$

solves the the Cauchy problem (3.1)*. This solution is the unique maximal solution to* (3.1)*, that is, if $v \in C^1((-\varepsilon, \varepsilon), \mathcal{H})$ solves* (3.1)*, then $v = u|_{(-\varepsilon,\varepsilon)}$.*

Proof. This proof is essentially the same as for linear ODEs.

To start with, we have

$$u(0) = e^0 u_0 = u_0. \tag{3.5}$$

We now show that $u \in C^1(\mathbb{R}, \mathcal{H})$ with derivative Au, so u solves (3.1). We have

$$\begin{aligned} \frac{u(t+h) - u(t) - hAu(t)}{h} &= \frac{1}{h} \sum_{j=0}^{\infty} \frac{(t+h)^j - t^j}{j!} A^j u_0 - A \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j u_0 \\ &= \frac{1}{h} \sum_{j=0}^{\infty} \frac{(t+h)^{j+1} - t^{j+1} - (j+1)ht^j}{(j+1)!} A^{j+1} u_0 \end{aligned} \tag{3.6}$$

The term with $j = 0$ vanishes, and for $j \geq 1$ we have by the mean value theorem

$$\begin{aligned} (t+h)^{j+1} - t^{j+1} - (j+1)ht^j &\overset{\tau_j \in [t,t+h]}{=} h(j+1)(\tau^j - t^j) \\ &\overset{\sigma_j \in [t,\tau_j]}{=} h\tau_j j(j+1)\sigma^{j-1}, \end{aligned} \tag{3.7}$$

so since $|\sigma_j|, |\tau_j| \leq |t| + |h|$, we have

$$\left| \frac{\mathrm{e}^{(t+h)A} u_0 - \mathrm{e}^{tA} u_0 - h}{h} \right| \leq |h| \|A\|^2 \mathrm{e}^{(|t|+|h|)\|A\|} \|u_0\|, \tag{3.8}$$

which converges to zero as $h \to 0$, so

$$\frac{\mathrm{d}\mathrm{e}^{At} u_0}{\mathrm{d}t} = A\mathrm{e}^{At} u_0, \tag{3.9}$$

which proves the claim. Since $u'(t) = \mathrm{e}^{At} A u_0$ has the same form, we can iterate this and obtain that $u \in C^\infty(\mathbb{R}, \mathcal{H})$.

Now assume that $v : (-\varepsilon, \varepsilon) \to \mathcal{H}$ solves (3.1). Then for all $|t| \leq \varepsilon$

$$\begin{aligned} \|u(t) - v(t)\|^2 &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \|u(s) - v(s)\|^2 \mathrm{d}s \\ &= \int_0^t 2\mathrm{Re}\langle (u(s) - v(s)), A(u(s) - v(s)) \rangle \mathrm{d}s \\ &\leq \int_0^t 2\|A\| \|u(s) - v(s)\|^2 \mathrm{d}s. \end{aligned} \tag{3.10}$$

Thus by Gronwall's inequality, this is less than the solution to the equation $x' = 2\|A\|x$, $x(0) = 0$, which vanishes. This proves uniqueness of u. □

Corollary 3.3. *For $t, s \in \mathbb{R}$ we have $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$*

Proof. While this can also be seen from the exponential series, it follows immediately from uniqueness of the solutions to (3.1) by the following argument. Let $f \in \mathcal{H}$ and consider the functions

$$\begin{aligned} u(t) &= \mathrm{e}^{(t+s)A} f \\ v(t) &= \mathrm{e}^{tA}\mathrm{e}^{sA} f. \end{aligned} \tag{3.11}$$

Both solve (3.1) with initial condition $v(0) = \mathrm{e}^{As} f = u(0)$, so they must be equal. Since f was arbitary this proves equality of the operators. □

3.2. The Hille-Yosida theorem

In this section we will prove a theorem that ensures the existence and uniqueness of solutions to the abstract Cauchy problem under suitable hypothesis on the generators A.

The key condition is that the generator should be (maximal) dissipative, which excludes directions of exponential growth for the solutions. With this condition the solutions will satisfy $\|u(t)\| \leq \|u_0\|$, instead of the general bound $\|u(t)\| \leq \mathrm{e}^{\|A\|}$, $A \in \mathrm{B}(\mathcal{H})$ which cannot be generalised to unbounded operators.

Definition 3.4. A densely defined operator A, $D(A)$ on a Hilbert space $\mathcal{H}$ is called *dissipative* if

$$\forall f \in D(A) : \mathrm{Re}\langle f, Af\rangle \leq 0. \tag{3.12}$$

The operator is called *maximal dissipative* if additionally $A - 1$ is surjective, i.e.

$$\mathrm{ran}(A-1) = \mathcal{H}. \tag{3.13}$$

Question 3.5. Which of the following operators with domain $D = H^2(\mathbb{R}^d)$ is dissipative?

1) $A_1 = \Delta$;

2) $A_2 = -\Delta$;

3) $A_3 = \mathrm{i}\Delta$.

Examples 3.6.

1) The operator $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ from Section 2.3 with $M : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ positive definite is dissipative since

$$\langle f, Af\rangle = -\int_{\mathbb{R}^d} \langle \nabla f(x), M(x)\nabla f(x)\rangle \mathrm{d}x \leq 0. \tag{3.14}$$

If M is uniformly elliptic (cf. (2.43)) and satisfies the hypothesis of Theorem 2.36 then it is maximal dissipative, since $\lambda - A$ is onto for all $\lambda > 0$ by Theorem 2.36.

2) If H, $D(H)$ is symmetric, then $A = \mathrm{i}H$ is dissipative, since

$$\langle f, Af\rangle = \mathrm{i}\langle f, Hf\rangle \in \mathrm{i}\mathbb{R}. \tag{3.15}$$

If H is self-adjoint, then by 2.49 A is maximal dissipative because

$$\mathrm{ran}(A-1) = \mathrm{ran}(\mathrm{i}H - 1) = \mathrm{ran}(H + \mathrm{i}) \overset{2.49}{=} \mathcal{H}. \tag{3.16}$$

Moreover, $-A$ is also maximal dissipative.

Proposition 3.7. *Let A, $D(A)$ be dissipative.*

a) For every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $A - z$ is injective;

b) If there exists $\lambda_0 > 0$ such that $A - \lambda_0$ is onto, then A is closed,

$$\sigma(A) \subset \{z \in \mathbb{C} : \mathrm{Re} z \leq 0\},$$

and for all z with $\mathrm{Re} z > 0$

$$\|R_z(A)\| \leq \frac{1}{\mathrm{Re} z}.$$

Proof. a) Let $z = \lambda + \mathrm{i}\mu$ with $\lambda > 0$. Clearly, $A - \mathrm{i}\mu$ is also dissipative, so it is sufficient to prove the statement for $\mu = 0$. We have for $f \in D(A)$, $\lambda > 0$

$$\begin{aligned}\|(A-\lambda)f\|^2 &= \|Af\|^2 + \lambda^2\|f\|^2 - 2\lambda \mathrm{Re}\langle f, Af, \rangle \\ &\geq \lambda^2\|f\|^2. \end{aligned} \tag{3.17}$$

This shows that $(A - \lambda)f = 0 \implies f = 0$, so $\ker(A - \lambda) = \{0\}$.

b) By a) and the fact that $A - \lambda_0$ is onto, we have that $A - \lambda_0$ is bijective. Applying (3.17) with $\lambda = \lambda_0$ and $f = (A - \lambda)^{-1} g$, $g \in \mathcal{H}$ we find

$$\|g\|^2 \geq \lambda_0^2 \|(A - \lambda_0)^{-1} g\|^2, \tag{3.18}$$

so $(A - \lambda_0)^{-1} \in \mathrm{B}(\mathcal{H})$ and $\lambda_0 \in \rho(A)$. By Problem 20, A is closed. We also have $\|R_{\lambda_0}(A)\| \leq \lambda_0^{-1}$, and by Theorem 2.56 (in particular (2.100)) we thus have $B(\lambda_0, \lambda_0) \subset \rho(A)$. Let $z = \lambda + \mathrm{i}\mu \in \rho(A)$ with $\lambda > 0$. Applying (3.17) to $A_\mu = A - \mathrm{i}\mu$ with $f = R_z(A) g$, $g \in \mathcal{H}$ we find

$$\|g\|^2 \geq \mathrm{Re}(z)^2 \|R_z(A) g\|^2. \tag{3.19}$$

Using this bound on the norm of the resolvent, we can then expand around additional points and enlarge the known resolvent set until it covers the right half plane. The spectrum is thus contained in the (closed) left half plane. □

Corollary 3.8. *Let A, $D(A)$ be dissipative. The following are equivalent*

1) A is maximal dissipative;

2) $A - z$ is surjective for all $z \in \mathbb{C}$ with $\mathrm{Re} z > 0$;

3) $A - \lambda$ is surjective for some $\lambda > 0$.

We will now work toward solving the abstract Cauchy problem (3.1) for a generator $A, D(A)$ that is maximal dissipative and an initial contition $u_0 \in D(A)$.

The idea is to use the spectral information on A we have obtained and approximate A by bounded operators, the so-called Yosida-approximants,

$$A_n := -nAR_n(A) \tag{3.20}$$

Lemma 3.9. *Let $A, D(A)$ be maximal dissipative and define A_n by (3.20) for $n \in \mathbb{N}$.*

a) $A_n \in \mathrm{B}(\mathcal{H})$ *and* $\|A_n\| \leq n$*;*

b) $nR_n(A) + n^{-1}A_n = -1$*;*

c) A_n *is dissipative;*

d) For all $f \in D(A)$ *we have* $\|A_n f\|_{\mathcal{H}} \leq \|Af\|_{\mathcal{H}}$*;*

e) For all $f \in \mathcal{H}$ *we have* $\lim_{n\to\infty} \|f + nR_n(A)f\|_{\mathcal{H}} = 0$*;*

f) For all $f \in D(A)$ *we have* $\lim_{n\to\infty} \|Af - A_n f\|_{\mathcal{H}} = 0$.

Proof. a) We have

$$A_n = -(A-n)nR_n(A) - n^2 R_n(A) = -n^2 R_n(A) - n. \tag{3.21}$$

Hence $A_n \in \mathrm{B}(\mathcal{H})$ and $\|A_n\| \leq 2n$ since $\|R_n(A)\| \leq n^{-1}$ by Proposition 3.7. The improved bound $\|A_n\| \leq n$ follows from c) and Problem 25.

b) This follows by dividing (3.21) by n.

c) By b) we have for $f \in \mathcal{H}$

$$\begin{aligned} \mathrm{Re}\langle f, A_n f\rangle &= \mathrm{Re}\langle -n^{-1}A_n - nR_n(A)f, A_n f\rangle \\ &= -n^{-1}\|A_n f\|^2 + n^2 \mathrm{Re}\langle R_n(A)f, AR_n(A)f\rangle \leq 0. \end{aligned} \tag{3.22}$$

d) Since $\|nR_n(A)\| \leq 1$ we have for $f \in D(A)$ (since $R_n(A)$ is both left and right inverse of $A - n$)

$$\|A_n f\| \overset{(3.21)}{=} \|n^2 R_n(A)f + nf\| = \|nR_n(A)Af\| \leq \|Af\|. \tag{3.23}$$

e) Let first $f \in D(A)$. Then by b)

$$\|f + nR_n(A)f\| = n^{-1}\|A_n f\| \leq n^{-1}\|Af\| \overset{n\to\infty}{\to} 0. \tag{3.24}$$

For the general case $f \in \mathcal{H}$ let $\varepsilon > 0$ and choose $g \in D(A)$ with $\|f - g\| < \varepsilon$ (using density of $D(A)$). Then

$$\|f + nR_n(A)f\| \leq \|g + nR_n(A)g\| + \|(1 + nR_n(A))(f-g)\| \leq n^{-1}\|Ag\| + 2\varepsilon, \tag{3.25}$$

so choosing n large enough this is less than 3ε, which proves the claimed convergence.

f) We have

$$\|Af - A_n f\| = \|(1 + nR_n(A))Af\|, \tag{3.26}$$

so the claim follows from e). $\square$

Theorem 3.10 (Hille-Yosida)**.** *Let $A, D(A)$ be maximal dissipative. For every $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,\infty), \mathcal{H}) \cap C([0,\infty), D(A)) \tag{3.27}$$

satisfying (3.1)*. Moreover, the map*

$$\begin{aligned} \Phi_A : D(A) &\to C([0,\infty), \mathcal{H}) \\ u_0 &\mapsto u \end{aligned}$$

is linear and satisfies

$$\|\Phi_A u_0\|_{C([0,\infty),\mathcal{H})} \leq \|u_0\|_{\mathcal{H}}.$$

It thus extends uniquely to a continuous map $\Phi_A : \mathcal{H} \to C([0,\infty), \mathcal{H})$ of norm one.

Proof. **Step 1 (uniqueness)** Assume u, $\tilde{u}$ are two solutions to (3.1). Then $u(0)-\tilde{u}(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u - \tilde{u}\|^2 = 2\mathrm{Re}\langle u - \tilde{u}, A(u - \tilde{u})\rangle \leq 0. \tag{3.28}$$

Thus $t \mapsto \|u(t) - \tilde{u}(t)\|$ is non-increasing, so

$$0 \leq \|u(t) - \tilde{u}(t)\| \leq \|u(0) - \tilde{u}(0)\| = 0$$

and the solution is unique.

Step 2 (approximate solutions u_n)

Let $u_n(t) = \Phi_{A_n} u(t) = \mathrm{e}^{A_n t} u_0$ be the unique solution of

$$\begin{cases} \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = A_n u_n\,, & t > 0 \\ u_n(0) = u_0 \end{cases} \tag{3.29}$$

(cf. Thm. 3.2).

First note that, since A_n is dissipative, we have for all $n \in \mathbb{N}$, $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2 = 2\mathrm{Re}\langle u_n(t), A_n u_n(t)\rangle \leq 0. \tag{3.30}$$

Consequently,

$$\|u_n(t)\| \leq \|u_0\|. \tag{3.31}$$

Since A_n is bounded, we may apply this reasoning with inital condition $v_0 = Au_0$ and obtain with Lemma 3.9d)

$$\|A_n u_n(t)\| = \|e^{A_n t} A_n u_0\| \leq \|A_n u_0\| \leq \|Au_0\|. \tag{3.32}$$

Step 3 (approximation of u) We now prove that u_n converges to some limit u uniformly on compact invervalls $[0, t_0] \subset [0,\infty)$.

Let $n, m \in \mathbb{N}$. Obviously we have $u_n(0) = u_0 = u_m(0)$, and by the fundamental theorem of calculus

$$\|u_n(t) - u_m(t)\|^2 = \int_0^t 2\mathrm{Re}\langle A_n u_n(s) - A_m u_m(s), u_n(s) - u_m(s)\rangle \mathrm{d}s. \tag{3.33}$$

Now by Lemma 3.9b), we have (for fixed s, which we drop from the notation)

$$\begin{aligned}
&\mathrm{Re}\langle A_n u_n - A_m u_m, u_n - u_m\rangle \\
&= \mathrm{Re}\langle A_n u_n - A_m u_m, -nR_n(A)u_n - n^{-1}A_n u_n + mR_m(A)u_m + m^{-1}A_m u_m\rangle \\
&= \underbrace{\mathrm{Re}\langle A(nR_n(A)u_n - mR_m(A)u_m), (nR_n(A)u_n - mR_m(A)u_m)\rangle}_{\leq 0}\rangle \\
&\qquad + \mathrm{Re}\langle A_n u_n - A_m u_m, -n^{-1}A_n u_n + m^{-1}A_m u_m\rangle \\
&\leq (\|A_n u_n\| + \|A_m u_m\|)(n^{-1}\|A_n u_n\| + m^{-1}\|A_m u_m\|).
\end{aligned} \tag{3.34}$$

By (3.32) we have for all $0 \leq t \leq t_0$

$$\|u_n(t) - u_m(t)\|^2 \leq \int_0^t 4\|Au_0\|^2(n^{-1} + m^{-1})\mathrm{d}s \leq 4t_0(n^{-1} + m^{-1})\|Au_0\|^2. \tag{3.35}$$

This proves that $u_n(t)$ is a Cauchy sequence for every t and thus converges to a limit $u(t) \in \mathcal{H}$. Since the bound above is uniform for $t \leq t_0$, the convergence is uniform and thus $t \mapsto u(t)$ is continuous, i.e.,

$$u \in C([0, \infty), \mathcal{H}). \tag{3.36}$$

Moreover, we have

$$\|u(t)\| = \lim_{n\to\infty} \|u_n(t)\| \leq \|u_0\|. \tag{3.37}$$

The map $u_0 \mapsto u$ is also linear since $u_0 \mapsto u_n$ is linear for every n. We have thus shown that $u_0 \mapsto \Phi_A u_0 = u$ is linear and bounded, and can thus be extended to $u_0 \in \mathcal{H}$.

Step 4 (differentiability) It remains to prove that, for $u_0 \in D(A)$, u is differentiable and a solution to (3.1). For $n \in \mathbb{N}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n(t) = A_n e^{A_n t} u_0 = e^{A_n t} A_n u_0. \tag{3.38}$$

Now

$$\|e^{A_n t} A_n u_0 - \Phi_A(t) A u_0\| \leq \|e^{A_n t}(A_n u_0 - A u_0)\| + \|(\Phi_A(t) - e^{A_n t})Au_0\|. \tag{3.39}$$

The first term converges to zero uniformly in t by (3.31) and Lemma 3.9f). The second term should converge to zero aswell since $u_n \to u$. However we have only proved this for $u_0 \in D(A)$ so far, and $Au_0 \notin D(A)$, in general. To close this gap, let $\varepsilon > 0$ and $v_0 \in D(A)$ with $\|v_0 - Au_0\| < \varepsilon$. Then

$$\|(\Phi_A(t) - \Phi_{A_n}(t))Au_0\| \leq \|(\Phi_A(t) - \Phi_{A_n}(t))v_0\| + \underbrace{\|(\Phi_A(t) - \Phi_{A_n}(t))(Au_0 - v_0)\|}_{<2\varepsilon}. \tag{3.40}$$

By the convergence of solutions with $u_0 \in D(A)$ proved above, we thus have for n large enough

$$\sup_{0\le t\le t_0} \|\Phi_{A_n}(t)A_n u_0 - \Phi_A(t)Au_0\| < 4\varepsilon. \tag{3.41}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}t}u_n$ converges uniformly to $\Phi_A Au_0$, which must then be equal to $\frac{\mathrm{d}}{\mathrm{d}t}u$, so

$$u \in C^1([0,\infty),\mathcal{H}). \tag{3.42}$$

Step 5 (u is a solution) From what we have proved so far, we know that for $t \ge 0$

$$\lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}u_n(t) = \lim_{n\to\infty} A_n u_n(t) = -\lim_{n\to\infty} AnR_n(A)u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t). \tag{3.43}$$

Using that $u_n(t)$ and $-nR_n(A)u(t)$ both converge to $u(t)$ (by Lemma 3.9e)) we have for fixed $t > 0$

$$\begin{aligned}\|nR_n(A)u_n(t) + u(t)\| &\le \|nR_n(A)(u_n(t) - u(t))\| + \|nR_n(A)u(t) + u(t)\| \\ &\le \underbrace{\|nR_n(A)\|}_{\le 1}\|u_n(t) - u(t)\| + \|nR_n(A)u(t) + u(t)\|,\end{aligned} \tag{3.44}$$

which tends to zero for $n \to \infty$. We thus have

$$\begin{aligned}-nR_n(A)u_n(t) &\to u(t)\\ -AnR_n(A)u_n(t) &\to \frac{\mathrm{d}}{\mathrm{d}t}u(t).\end{aligned} \tag{3.45}$$

Since A is closed by Proposition 3.7, this implies that $u(t) \in D(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = Au(t)$, i.e., u is indeed a solution to (3.1). Moreover, since Au equals the (continuous) derivative of u, we have

$$u \in C([0,\infty), D(A)). \tag{3.46}$$

This completes the proof.

□

Remark 3.11. The Theorem says that the abstract Cauchy problem (3.1) is *well posed* (in the sense of Hadamard) in $D(A)$, that is, we have

1. Existence of a solution for every initial datum,
2. Uniqueness of this solution,
3. Continuous dependence of the solution on the initial data (by boudedness of Φ_A).

Let A, $D(A)$ be maximal dissipative and $t \ge 0$. We define a bounded operator e^{tA} on $\mathcal{H}$ by

$$\mathrm{e}^{At}f := (\Phi_A f)(t),$$

i.e., $\mathrm{e}^{At}f$ is the solution to the Cauchy problem (3.1) with initial condition f evaluated at time t.

We caution that e^{At} is in general not given by the exponential series, which might not converge. Moreover, it is only defined for $t \geq 0$, since Theorem 3.10 only gives existence of solutions for positive time. It could be defined for $t \leq 0$ if $-A$ is maximal dissipative.

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i.e., $e^{At} f$ is the solution to the Cauchy problem (3.1) with initial condition f evaluated at time t.

We caution that e^{At} is in general not given by the exponential series, which might not converge. Moreover, it is only defined for $t \geq 0$, since Theorem 3.10 only gives existence of solutions for positive time. It could be defined for $t \leq 0$ if $-A$ is maximal dissipative.

Corollary 3.12. *Let A, $D(A)$ be maximal dissipative and $t, s \geq 0$, then*

$$e^{(t+s)A} = e^{tA} e^{sA}. \tag{3.47}$$

Proof. This follows from uniqueness of solutions as in Corollary 3.3. □

Remark 3.13. The existence result we have proved is actually optimal, in the following sense.

Assume that the family of bounded operators $T(t)$, $t \geq 0$, forms a *strongly continuous semi-group*, that is we have

1) $T(0) = 1$,

2) $T(t+s) = T(t)T(s)$,

3) $\forall f \in \mathcal{H} : \lim_{t \to 0} T(t) f = f$.

Assume moreover that $\|T(t)\| \leq 1$ for all $t \geq 0$. Then there exists a maximal dissipative operator $A, D(A)$ so that $T(t) = e^{tA}$, see [RS2, Thm.X.47a].

It is easy to generalise this result to operators with spectrum in the half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq \mu\}$ satisfying

$$\mathrm{Re}\langle f, Af \rangle \leq \mu \|f\|^2. \tag{3.48}$$

for some $\mu \in \mathbb{R}$. By Proposition 3.7 then $A - \mu$ is maximal dissipative. It is clear that u is a solution to (3.1) if and only if

$$u(t) = e^{\mu t} e^{t(A-\mu)} u_0. \tag{3.49}$$

Corollary 3.14. *Let A, $D(A)$ be densely defined, satisfy* (3.48) *for some $\mu \in \mathbb{R}$ and assume that $A - z$ is onto for some z with* $\mathrm{Re} z > \mu$. *Then* (3.49) *is the unique solution to the Cauchy Problem* (3.1). *This solution satisfies*

$$\|u(t)\|_{\mathcal{H}} \leq e^{\mu t} \|u_0\|_{\mathcal{H}} \tag{3.50}$$

3.3. Applications of the Hille-Yosida theorem

3.3.1. The Schrödinger equation

We will now study the Cauchy problem for the Schrödinger equation with potential

$$\begin{cases} \mathrm{i}\dfrac{\mathrm{d}u}{\mathrm{d}t} = -\Delta u + Vu\,, & t > 0, \\ u(0) = u_0 \end{cases} \tag{3.51}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a real valued potential that acts as an operator of multiplication. Since the quantity $|u(t,x)|^2$ represents the probabilty density for the position of a particle (or the configuration of a system) at time t, we additionally want that

$$\int |u(t,x)|^2 \mathrm{d}x = \|u(t)\|^2 = 1 \tag{3.52}$$

holds for all t. This is equivalent to self-adjointness of the operator $H = -\Delta + V$ on a suitable domain. For this case, we have as a Corollary to the Hille-Yosida theorem

Corollary 3.15. *Let $H, D(H)$ be self-adjoint and $u_0 \in D(H)$. Then there exists a unique solution*

$$u \in C(\mathbb{R}, D(H)) \cap C^1(\mathbb{R}, \mathcal{H}) \tag{3.53}$$

of the Cauchy problem

$$\begin{cases} \mathrm{i}\dfrac{\mathrm{d}u}{\mathrm{d}t} = Hu\,, & t > 0 \\ u(0) = u_0 \end{cases} \tag{3.54}$$

Moreover, the solution operator $\mathrm{e}^{-\mathrm{i}tH}$ *is unitary.*

Proof. Existence and uniquencess for $t \geq 0$ follows from the Hille-Yosida theorem, since $A = -\mathrm{i}H$ is maximal dissipative. For $t \leq 0$, consider $v(t) = u(-t)$, which should solve

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(-t) = \mathrm{i}Hv(t). \tag{3.55}$$

Since $A = \mathrm{i}H$ is also maximal dissipative, existence and uniqueness of v follows.

The solution operator $\mathrm{e}^{-\mathrm{i}tH} \in \mathrm{B}(\mathcal{H})$ is thus defined for every $t \in \mathbb{R}$. Moreover, for $u_0 \in D(H)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\mathrm{Re}\langle u(t), -\mathrm{i}Hu(t)\rangle = 0, \tag{3.56}$$

so

$$\|\mathrm{e}^{-\mathrm{i}tH}u_0\| = \|u_0\|. \tag{3.57}$$

The unique extension of $\mathrm{e}^{-\mathrm{i}tH} := U$ thus satisfies $U^*U = 1$. Now let $u, v \in D(H)$ and define $u(t) := (\mathrm{e}^{-\mathrm{i}tH})^* u$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle u, \mathrm{e}^{-\mathrm{i}tH}v\rangle = \langle u, -\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}Hv\rangle = \langle \mathrm{i}Hu(t), v\rangle. \tag{3.58}$$

Hence $(\mathrm{e}^{-\mathrm{i}tH})^* = \mathrm{e}^{\mathrm{i}tH}$, since $\|\mathrm{e}^{\mathrm{i}tH}u\| = \|u\|$ we have $UU^* = 1$ and $\mathrm{e}^{-\mathrm{i}tH}$ is unitary. □

Our task is thus to find hypothesis on V that ensure self-adjointness of $-\Delta + V$ on a suitable domain – usually $H^2(\mathbb{R}^d)$. The following proposition is extremely useful for studying operators that can be decomposed into a dominant part (usually the part with the highest order derivativies) which is known to be maximal dissipative (self-adjoint), and a secondary part which can be bounded by the dominant one.

Proposition 3.16. *Let A, $D(A)$ be a maximal dissipative operator and B, $D(B)$ dissipative with $D(A) \subset D(B)$. Assume that there exists $0 \le \varepsilon < 1$ and $C > 0$ so that for all $f \in D(A)$*

$$\|Bf\| \le \varepsilon \|Af\| + C\|f\|.$$

Then $A + B$ with domain $D(A + B) = D(A)$ is maximal dissipative.

Proof. First note that $A + B$ is defined on $D(A)$ and for $f \in D(A)$

$$\mathrm{Re}\langle f, (A + B)f\rangle = \mathrm{Re}\langle f, Af\rangle + \mathrm{Re}\langle f, Bf\rangle \le 0, \tag{3.59}$$

since both are dissipative. Hence $A + B$ is densely defined and dissipative. By Corollary 3.8 it is now sufficent to prove that $A + B - \lambda$ is onto for some $\lambda > 0$.

Using the hypothesis, Proposition 3.7 and Exercise 10.1 we obtain

$$\|BR_\lambda(A)\| \le \varepsilon \|AR_\lambda(A)\| + C \|R_\lambda(A)\| \le \varepsilon + \frac{C}{\lambda}. \tag{3.60}$$

If $\varepsilon + C/\lambda < 1$, the bounded operator $1 + BR_\lambda(A)$ is thus invertible by a Neumann series. Since $A - \lambda$ is onto, then so is

$$(1 + BR_\lambda(A))(A - \lambda) = A + B - \lambda. \tag{3.61}$$

This completes the proof. □

Corollary 3.17 (Kato-Rellich)**.** *Let H, $D(H)$ be self-adjoint and K, $D(K)$ symmetric with $D(H) \subset D(K)$. Assume that there exists $0 \le \varepsilon < 1$ and $C > 0$ so that for all $f \in D(H)$*

$$\|Kf\| \le \varepsilon \|Hf\| + C\|f\|.$$

Then $H + K$ with domain $D(H + K) = D(H)$ is self-adjoint.

Proof. Apply Proposition 3.16 to $A = \mathrm{i}H$, $B = \mathrm{i}K$. □

Examples 3.18.

a) Let $V \in L^\infty(\mathbb{R}, \mathbb{R})$. Then the hypothesis of Corollary 3.17 are satisfied with $\varepsilon = 0$ and $C = \|V\|_\infty$. Hence $H = -\Delta + V$ is self-adjoint on $D(H) = H^2(\mathbb{R}^d)$. We can thus solve the Schrödinger equation for every bounded potential.

b) Let $d \leq 3$ and $V \in L^2(\mathbb{R}^d, \mathbb{R})$. By Sobolev's Lemma we know that $H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $d \leq 3$. We can bound

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq \|V\|_{L^2}\|f\|_{L^\infty}. \tag{3.62}$$

Moreover, proceeding as in the proof of Sobolev's Lemma, we have for $\delta > 0$

$$\|f\|_\infty \leq (2\pi)^{-d/2}\|\hat{f}\|_{L^1} \leq (2\pi)^{-d/2}\|(1+\delta p^2)\hat{f}\|_{L^2}\|(1+\delta p^2)^{-1}\|_{L^2}. \tag{3.63}$$

Since

$$\|(1+\delta p^2)^{-1}\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(1+\delta p^2)^2} = \delta^{-d/2} \int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(1+p^2)^2}, \tag{3.64}$$

we thus have

$$\|Vf\|_{L^2(\mathbb{R}^d)} \leq C(\delta^{1-d/4}\|-\Delta f\| + \delta^{-d/4}\|f\|_{L^2}), \tag{3.65}$$

with

$$C = (2\pi)^{-d/2}\|V\|_{L^2}\|(1+p^2)^{-1}\|_{L^2}^2. \tag{3.66}$$

Since $1 - d/4 > 0$ we can make the constant in front of $\|-\Delta f\|$ as small as we wish, and the hypothesis of Corollary 3.17 are satisfied. We can thus solve Schrödinger's equation with $V \in L^2(\mathbb{R}^d)$ if $d \leq 3$.

c) Let $d = 3$ and consider the Hamiltonian for the electron in the Hydrogen atom

$$H = -\Delta - \frac{\alpha}{|x|} \tag{3.67}$$

with $D(H) = H^2(\mathbb{R}^3)$. The Coulomb potential can be written as

$$\frac{1}{|x|} = \underbrace{\frac{1(|x| \leq 1)}{|x|}}_{\in L^2(\mathbb{R}^3)} + \underbrace{\frac{1(|x| > 1)}{|x|}}_{\in L^\infty(\mathbb{R}^3)}. \tag{3.68}$$

By example b) above,

$$H_1 := -\Delta - \alpha\frac{1(|x| \leq 1)}{|x|} \tag{3.69}$$

is self-adjoint on $H^2(\mathbb{R}^3)$. By example a) above, after adding a bounded potential we still have a self-adjoint operator with the same domain, so H is self-adjoint.

3.3.2. The heat equation

We will now apply the Hille Yosida theorem to the heat equation with variable coefficients.

$$\begin{cases} \partial_t u(t,x) = \operatorname{div} M(x)\nabla u(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{3.70}$$

This equation cannot be solved directly via the Fourier transform if the coefficient matrix M is not constant. This matrix M models material properties of the medium through which the heat is flowing, such as regions of better conductivity or different conductivity depending on the direction.

Existence of solutions follows from Theorem 3.10 and Theorem 2.36.

Corollary 3.19. *Let $M \in C^1(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ be bounded with bounded derivatives and satisfy for $a > 0$*

$$\forall v \in \mathbb{C}^d, x \in \mathbb{R}^d : \langle v, M(x)v \rangle \geq a\|v\|^2. \tag{3.71}$$

Let $u_0 \in H^2(\mathbb{R}^d)$. The equation

$$\begin{cases} \partial_t u = \operatorname{div} M(x)\nabla u \qquad t \geq 0 \\ u(0,x) = u_0(x) \end{cases} \tag{3.72}$$

has a unique solution

$$u \in C^1\Big([0,\infty), L^2(\mathbb{R}^d)\Big) \cap C\Big([0,\infty), H^2(\mathbb{R}^d)\Big). \tag{3.73}$$

Proof. We need to show that the operator $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ satisfies the hypothesis of Theorem 3.10, i.e. A is maximal dissipative. First, A is dissipative, since

$$\langle f, \operatorname{div} M\nabla f\rangle = -\int_{\mathbb{R}^d} \langle M(x)\nabla f(x), \nabla f(x)\rangle \mathrm{d}x \leq -a\int |\nabla f(x)|^2 \mathrm{d}x < 0. \tag{3.74}$$

To show that A is maximal disspipative, we need to prove that $A - 1$ is onto. This follows from Theorem 2.36, as this states that the equation

$$(\lambda - A)u = f \tag{3.75}$$

has a unique solution $u \in H^2(\mathbb{R}^d) = D(A)$ for all $f \in L^2(\mathbb{R}^d)$. □

While this first result is a direct corollary of the Hille-Yosida theorem, we can obtain stronger statetements using more specific properties of the equation. The heat equation belongs to the class of PDEs called *parabolic*. The most important feature of these is that the solution is more regular at positive times than at $t = 0$, i.e., they are smoothing. Apart from generalized (linear) heat equations the class of parabolic equations also contains the non-linear Ricci flow that was used in Perelman's proof of the Poincaré conjecture. The smoothing property gives rise to a preferred direction of time and a form of irreversible behaviour.

The abstract property giving rise to this behaviour is that the generator A, $D(A)$ in the Cauchy problem is

1. self-adjoint,

2. non-positive (dissipative).

Together, these imply that $\sigma(A) \subset (-\infty, 0]$. If we want to solve the equation for negative times, we can consider $v(t) = u(-t)$ for $t > 0$. This would solve the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = -\frac{\mathrm{d}u}{\mathrm{d}t}(-t) = -Av \tag{3.76}$$

Note that if $\sigma(A)$ is unbounded, then $-A$ cannot satisfy the hypothesis of corollary 3.14 for any $\mu > 0$. We thus cannot say that the evolution exists for negative times, which is in accordance with the idea that the heat flow is irreversible.

The following theorem makes precise the smoothing for a abstract form of "heat equation".

Proposition 3.20. *Let A, $D(A)$ be a self-adjoint and non-positive operator on $\mathcal{H}$. Let $u_0 \in \mathcal{H}$ and define*

$$u(t) := \mathrm{e}^{At}u_0 \in C([0,\infty),\mathcal{H}).$$

Then

$$u \in C((0,\infty), D(A)) \cap C^1((0,\infty),\mathcal{H}) \tag{3.77}$$

and u solves the equation (3.1) *for $t > 0$.*

Proof. The strategy is to assume first that $u_0 \in D(A)$ and to prove that

$$\mathrm{e}^{At} : D(A) \to C^1([\varepsilon,\infty),\mathcal{H}) \tag{3.78}$$

is bounded with respect to the norm of $\mathcal{H}$ for any $\varepsilon > 0$. Then we can again extend it to $\mathcal{H}$ and by uniqueness of the extension this will prove the claim.

Let $u_0 \in D(A)$ and $u(t) = \mathrm{e}^{At}u_0$. Let A_n be the Yosida approximants (3.20) and $u_n(t) := \mathrm{e}^{A_n t}u_0$ be the correspondign solutions to (3.29). Note that A_n is bounded and self-adjoint. Recall that $\|\mathrm{e}^{A_n t}\| \le 1$, so in particular

$$\|A_n u_n(t)\| = \|\mathrm{e}^{A_n(t-s)}A_n u_n(s)\| \le \|A_n u_n(s)\| \tag{3.79}$$

for any $0 \le s \le t$. Consequently,

$$\int_0^T t\|A_n u_n(t)\|^2 \mathrm{d}t \ge \|A_n u_n(T)\|^2 \int_0^T t \mathrm{d}t = \tfrac{1}{2}T^2\|A_n u_n(T)\|^2. \tag{3.80}$$

Note that the right hand side is the object we want to control. On the other hand, by self-adjointness of A_n, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle A_n u_n, u_n\rangle = \langle A_n^2 u_n, u_n\rangle + \langle A_n u_n, A_n u_n\rangle = 2\|A_n u_n\|^2. \tag{3.81}$$

After integration by parts, this gives

$$\begin{aligned}
\int_0^T t\|A_n u_n(t)\|^2 \mathrm{d}t &= \frac{1}{2}\int_0^T t\frac{\mathrm{d}}{\mathrm{d}t}\langle A_n u_n(t), u_n(t)\rangle \mathrm{d}t \\
&= \frac{T}{2}\langle A_n u_n(T), u_n(T)\rangle - \frac{1}{2}\int_0^T \langle A_n u_n(t), u_n(t)\rangle \mathrm{d}t \\
&= \tfrac{1}{2}T\langle A_n u_n(T), u_n(T)\rangle - \tfrac{1}{4}\Big(\|u_n(T)\|^2 - \|u_0\|^2\Big) \\
&\le \tfrac{1}{2}T\|A_n u_n(T)\|\|u_0\| + \tfrac{1}{4}\|u_0\|^2 \\
&\le \tfrac{1}{4}T^2\|A_n u_n(T)\|^2 + \tfrac{1}{2}\|u_0\|^2.
\end{aligned} \tag{3.82}$$

With (3.80) this gives

$$T\|\tfrac{\mathrm{d}u_n}{\mathrm{d}t}(T)\| = T\|A_n u_n(T)\| \le \sqrt{2}\|u_0\|. \tag{3.83}$$

As $\frac{\mathrm{d}u_n}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t}$ (see step 4 of the proof of Theorem 3.10) and $A_n u_n \to Au$ (see step 5 of the proof of Theorem 3.10), passing to the limit $n \to \infty$ proves that

$$\sup_{t\geq\varepsilon} \|Au(t)\|_{\mathcal{H}} \leq \sqrt{2}\varepsilon^{-1}\|u_0\|_{\mathcal{H}} \tag{3.84}$$

$$\sup_{t\geq\varepsilon} \|\tfrac{\mathrm{d}u_n}{\mathrm{d}t}(t)\|_{\mathcal{H}} \leq \sqrt{2}\varepsilon^{-1}\|u_0\|_{\mathcal{H}}, \tag{3.85}$$

so Φ_A has norm less than $\sqrt{2}\varepsilon^{-1}$ as a map from $D(A) \subset \mathcal{H}$ to $C([\varepsilon,\infty), D(A))$ and $C^1([\varepsilon,\infty),\mathcal{H})$. It thus extends to $u \in \mathcal{H}$ by the B.L.T. theorem, so by choosing $\varepsilon = t/2$ we see that $\mathrm{e}^{At}u_0$ is differentiable at every $t > 0$ and the equation (3.1) holds by continuity since it holds for the approximants. □

Corollary 3.21. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 3.19. Let $u_0 \in L^2(\mathbb{R}^d)$, then $u(t) = \mathrm{e}^{At}u_0$ satisfies*

$$u \in C((0,\infty), H^2(\mathbb{R}^d)) \cap C^1((0,\infty), L^2(\mathbb{R}^d))$$

and u solves the heat equation (3.70) *for $t > 0$.*

Lemma 3.22. *Let A, $D(A)$ be self-adjoint on $\mathcal{H}$ and define*

$$D(A^2) := \{f \in D(A) : Af \in D(A)\}.$$

Then $D(A^2)$ is dense in $D(A)$ with the graph norm as well as $\mathcal{H}$ and the operator $B = A$, $D(B) = D(A^2)$ is self-adjoint on the Hilbert space $D(A)$ with the scalar product

$$\langle f, g\rangle_{D(A)} := \langle f, g\rangle_{\mathcal{H}} + \langle Af, Ag\rangle_{\mathcal{H}}.$$

Proof. Let $z \in \rho(A)$ (for example $z = \pm\mathrm{i}$) and $f \in D(A)$. Then $R_z(A)f \in D(A^2)$ since

$$AR_z(A)f = R_z(A)Af \in D(A). \tag{3.86}$$

Since $D(A)$ is dense in $\mathcal{H}$ and $R_z(A) : \mathcal{H} \to D(A)$ is continuous and surjective, we can thus approximate every element of $g = R_z(A)f \in D(A)$ by $R_z(A)f_n \in D(A^2)$, i.e, $D(A^2) \subset D(A)$ is dense. As the inclusion of $D(A)$ in $\mathcal{H}$ is dense and continuous we can also approximate every $f \in \mathcal{H}$ in $D(A^2)$.

Obviously, $B = A : D(A^2) \to D(A)$ is symmetric. Moreover, the map $R_z(A) : D(A) \to D(A^2)$ is the inverse of $B - z$, so in particular $\pm\mathrm{i} \in \rho(B)$ and thus, by the criterion of Theorem 2.49, B is self-adjoint. □

In view of this lemma we iteratively define

$$D(A^k) = \{f \in D(A^{k-1}) : Af \in D(A^{k-1})\}, \tag{3.87}$$

and obtain that for $\ell < k$, $D(A^k) \subset D(A^\ell)$ is dense and the restriction of A to $D(A^k)$ is self adjoint on $D(A^{k-1})$, with the convention $D(A^0) = \mathcal{H}$.

Theorem 3.23. *Let A, $D(A)$ be a self-adjoint and dissipative operator on $\mathcal{H}$. Let $u_0 \in \mathcal{H}$ and u be the solution to* (3.1) *given by Proposition 3.20. Then for all $k \in \mathbb{N}_0$*

$$u \in C^\infty((0,\infty), D(A^k)). \tag{3.88}$$

Proof. Let $t_0 > 0$. By Proposition 3.20, $u(t_0) \in D(A)$. We now consider the equation

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}s}(s) &= Bv \qquad t > 0 \\ v(0) &= u(t_0). \end{aligned} \tag{3.89}$$

on $\mathcal{H} = D(A)$, where $B = A : D(A^2) \to D(A))$ is the restriction of A to $D(A^2)$. By Lemma 3.22, B is self-adjoint and since it restricts A it is also dissipative. We may thus apply Proposition 3.20 to obtain a solution to this equation. Clearly, we have $v(t) = u(t_0 + t)$ by uniqueness of the solution of 3.20. Thus, we have that

$$u \in C((t_0,\infty), D(A^2)) \cap C^1((t_0,\infty), D(A)). \tag{3.90}$$

Then $\frac{\mathrm{d}u}{\mathrm{d}t} = Au \in C^1((t_0,\infty),\mathcal{H})$ and thus also

$$u \in C^2((t_0,\infty),\mathcal{H}), \tag{3.91}$$

for any $t_0 > 0$. This shows that

$$u \in C^\ell((0,\infty), D(A^k)) \tag{3.92}$$

for $k + \ell \leq 2$. Iterating this argument yields the same for any $k, \ell \in \mathbb{N}_0$, which proves the claim. □

Corollary 3.24. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 3.19 and assume additionally that $M \in C^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d))$ with bounded derivatives. Let $u_0 \in L^2(\mathbb{R}^d)$, then $u(t) = \mathrm{e}^{tA}u_0$ satisfies*

$$u \in C^\infty((0,\infty) \times \mathbb{R}^d)$$

and u is a classical solution to the heat equation (3.70) *for $t > 0$.*

Proof. In view of Theorem 3.23 it is sufficent to prove that $C^\ell(\mathbb{R}^d) \subset D(A^k)$ for some k. This will follow from Sobolev's Lemma once we prove that $D(A^k) = H^{2k}(\mathbb{R}^d)$ in the lemma below. □

Lemma 3.25. *Let $A = \operatorname{div} M\nabla$, $D(A) = H^2(\mathbb{R}^d)$ be as in Corollary 3.19 and assume additionally that $M \in C^\infty(\mathbb{R}^d, \mathrm{B}(\mathbb{C}^d)$ with bounded derivatives. Then for $k \in \mathbb{N}$*

$$D(A^k) = H^{2k}(\mathbb{R}^d). \tag{3.93}$$

Proof. Assume that $f \in D(A^2)$, that is, $f \in D(A) = H^2(\mathbb{R}^d)$ and

$$(\lambda - A)f = g \in H^2(\mathbb{R}^d), \tag{3.94}$$

for $\lambda > 0$. Then for all $\varphi \in H^2(\mathbb{R}^d)$

$$\langle g, \varphi \rangle = \langle (\lambda - A)f, \varphi \rangle = \lambda \langle f, \varphi \rangle + \langle M\nabla f, \nabla \varphi \rangle, \tag{3.95}$$

i.e., f is a weak solution to the equation $\lambda f - \operatorname{div} M\nabla f = g$. Now let $j \in \{1, \dots, d\}$, and consider

$$\begin{aligned} \langle M\nabla \partial_j f, \nabla \varphi \rangle &= \langle \partial_j M\nabla f, \nabla \varphi \rangle - \langle (\partial_j M)\nabla f, \nabla \varphi \rangle \\ &= -\langle M\nabla f, \nabla \partial_j \varphi \rangle - \langle (\partial_j M)\nabla f, \nabla \varphi \rangle. \end{aligned} \tag{3.96}$$

Choosing $\varphi \in H^3(\mathbb{R}^d)$ we can use the equation and obtain

$$\begin{aligned} \lambda \langle \partial_j f, \varphi \rangle + \langle M\nabla \partial_j f, \nabla \varphi \rangle &= -\underbrace{\left(\lambda \langle f, \partial_j \varphi \rangle + \langle M\nabla f, \nabla \partial_j \varphi \rangle\right)}_{=\langle g, \partial_j \varphi \rangle} - \langle (\partial_j M)\nabla f, \nabla \varphi \rangle \\ &= \langle \partial_j g, \varphi \rangle + \langle \operatorname{div}(\partial_j M)\nabla f, \varphi \rangle. \end{aligned} \tag{3.97}$$

Thus $\partial_j f$ is a weak solution to the equation

$$\lambda u - \operatorname{div} M\nabla u = \partial_j g + \operatorname{div}\left((\partial_j M)\nabla f\right) \in L^2(\mathbb{R}^d), \tag{3.98}$$

where the right hand side is in L^2 because $g \in D(A) = H^2$ and $f \in H^2$. By Theorem 2.36 we thus have $\partial_j f \in H^2$ and thus $f \in H^3(\mathbb{R}^d)$.

To arrive at $f \in H^4(\mathbb{R}^d)$ we consider the second derivatives. Let $i, j \in \{1, \dots, d\}$. Then by the same argument as above, $\partial_{ij} f$ is a weak solution to

$$\lambda u - \operatorname{div} M\nabla u = \partial_{ij} g + \operatorname{div}\left((\partial_{ij} M)\nabla f\right) + \operatorname{div}\left((\partial_i M)\nabla \partial_j f\right) + \operatorname{div}\left((\partial_j M)\nabla \partial_i f\right). \tag{3.99}$$

The right hand side is in L^2 since $f \in H^3$ and the derivatives of M are bounded. Hence again by Theorem 2.36 we have $\partial_{ij} f \in H^2(\mathbb{R}^d)$ and thus $f \in H^4(\mathbb{R}^d)$.

It remains to prove the claim for $k > 2$. We proceed by induction, so assume that $D(A^{k-1}) = H^{2k-2}(\mathbb{R}^d)$ holds for $k \leq \ell$. For $f \in D(A^\ell) \subset D(A^{\ell-1})$ we then know that $f \in H^{2\ell-2}(\mathbb{R}^d)$. Consequently, the expression $\operatorname{div} M\nabla$ acting on $\partial_j f$ equals the operator A, and we have

$$(\lambda - A)\partial_j f = \partial_j g + \operatorname{div}\left((\partial_j M)\nabla f\right) \in H^{2\ell-4} = D(A^{\ell-2}). \tag{3.100}$$

From this we conclude that $\partial_j f \in D(A^{\ell-1}) = H^{2\ell-2}$ and thus $f \in H^{2\ell-1}(\mathbb{R}^d)$. The same reasoning for $\partial_{ij} f$ then shows that

$$(\lambda - A)\partial_{ij} = \partial_{ij} g + \operatorname{div}\left((\partial_{ij} M)\nabla f\right) + \operatorname{div}\left((\partial_i M)\nabla \partial_j f\right) + \operatorname{div}\left((\partial_j M)\nabla \partial_i f\right), \tag{3.101}$$

where the right hand side is an element of $H^{2\ell-1-3} = D(A^{\ell-2})$. We thus have $\partial_{ij} \in D(A^{\ell-1}) = H^{2\ell-2}$ and $f \in H^{2\ell}(\mathbb{R}^d)$. This completes the proof.

□

A. Problems

Problem 1. Use the Fourier transform to find a solution to the equation

$$\begin{cases} \partial_t u(t,x) - \partial_x u(t,x) = 0 \\ \qquad\qquad\qquad u(0,x) = u_0(x). \end{cases}$$

with $u_0 \in \mathscr{S}(\mathbb{R})$.

Problem 2 (The Fourier transform of complex Gaussians)**.**

a) Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. Show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}}.$$

b) Calculate the Fourier transform of $f(x) = \mathrm{e}^{-ax^2}$ for $\mathrm{Re}(a) > 0$.
Hint: Use Cauchy's theorem from complex analysis.

Problem 3 (The free Schrödinger equation)**.** In this exercise we show that the free Schrödinger equation in one dimension

$$\mathrm{i}\partial_t u(t,x) = -\tfrac{1}{2}\partial_x^2 u(t,x)$$

is solved for $t > 0$ by

$$u(t,x) = \frac{1}{\sqrt{2\pi \mathrm{i}t}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{(x-y)^2}{2t}} u_0(y)\mathrm{d}y,$$

for any $u_0 \in \mathscr{S}(\mathbb{R})$.

a) Show that for $(t,x) \in (0,\infty) \times \mathbb{R}$, $u(t,x)$ is continuously differentiable in t and twice continuously differentiable in x.

b) Show that $u(t,x)$ solves the Schrödinger equation for $(t,x) \in (0,\infty) \times \mathbb{R}$.

Problem 4 (Multiplication and convolution on $\mathscr{S}'$)**.** Let $g \in \mathscr{S}(\mathbb{R}^d)$ and define the multiplication by g as a map $(M_g f)(x) := g(x)f(x)$.

a) Show that $M_g' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is linear and continuous;

b) For $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ define multiplication with g by $g\varphi := M_g'\varphi$ and show that

$$\mathscr{F}(g\varphi) = (2\pi)^{-d/2}\hat{g} * \hat{\varphi},$$

where $*$ is the convolution of $\hat{g} \in \mathscr{S}(\mathbb{R}^d)$ with $\hat{\varphi} \in \mathscr{S}'(\mathbb{R}^d)$ defined in the lecture.

Problem 5 (The Fourier transform of complex Gaussians II)**.** Let $t > 0$ and set for $\varepsilon \geq 0$

$$f_\varepsilon(x) := \mathrm{e}^{-(\mathrm{i}t+\varepsilon)x^2/2}.$$

a) Show that $f_\varepsilon \to f_0$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \to 0$;

b) Show that

$$\hat{f}_0(p) = \frac{\mathrm{e}^{\mathrm{i}\frac{p^2}{2t}}}{\sqrt{\mathrm{i}t}}.$$

Can you explain the relation to Problem 3?

Hint: We know from Problem 2 that for $\varepsilon > 0$

$$\hat{f}_\varepsilon(p) = \frac{\mathrm{e}^{-\frac{p^2}{2(\mathrm{i}t+\varepsilon)}}}{\sqrt{\mathrm{i}t+\varepsilon}}.$$

Problem 6 (The delta-distribution)**.** Define for $f \in \mathscr{S}(\mathbb{R}^d)$

$$\delta_0(f) := f(0).$$

Let $g \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g = 1$ and define, for $\epsilon > 0$, $g_\epsilon(x) := \epsilon^{-d} g(\epsilon^{-1}x)$. Prove:

a) For every $\epsilon > 0$, g_ϵ defines a tempered distribution $\varphi_\epsilon \in \mathscr{S}'(\mathbb{R}^d)$ via

$$\varphi_\epsilon(f) := \int_{\mathbb{R}^d} \overline{g_\epsilon(x)} f(x)\mathrm{d}x, \qquad f \in \mathscr{S}(\mathbb{R}^d);$$

b) As $\epsilon \to 0$, $\varphi_\epsilon \to \delta_0$ in $\mathscr{S}'(\mathbb{R}^d)$;

c) Let $\theta \in \mathscr{S}'(\mathbb{R})$ be defined by $\theta(f) := \int_{\mathbb{R}} 1_{[0,\infty)}(x) f(x)\mathrm{d}x$. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\theta = \delta_0$. (Here $\frac{\mathrm{d}}{\mathrm{d}x} := (\partial^1)_{\mathscr{S}'}$ is the distributional derivative as defined in the lecture).

Problem 7. Let $k \in \mathbb{N}_0$, $f \in H^k(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$. Prove that $fg \in H^k(\mathbb{R}^d)$ and the generalised Leibniz rule holds for the derivatives of order $|\alpha| \leq k$.

Problem 8. Let X be a normed space and $S \subset X$. A point $x \in X$ is called a limit point of S if there exists a sequence in $x_n \in S$, $n \in \mathbb{N}$ that converges to x. Prove that

a) S is closed if and only if it contains all its limit points;

b) The closure $\overline{S}$ is the union of S and its limit points;

c) Assume that X is complete and $Y \subset X$ a subspace. Then Y is complete if and only if Y is closed.

Problem 9 (Green's function for the Laplacian)**.**

a) Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. Show that g (more precisely the associated distribution, φ_g) is the unique solution in $\mathscr{S}'(\mathbb{R})$ to the equation

$$(1 - \Delta)\varphi = \delta_0 ,$$

by

1) the Fourier transform;

2) directly using the distributional derivative.

b) Prove that for $f \in \mathscr{S}(\mathbb{R})$ the unique solution to the equation

$$(1 - \Delta)u = f$$

is

$$u(x) = \int g(x - y)f(y)\mathrm{d}y.$$

Remark: g is called the fundamental solution or Green's function for the equation.

Problem 10. Let $\mathcal{H}$ be a Hilbert space, and $S \subset \mathcal{H}$ a subset.

a) Show that $\overline{\operatorname{span}(S)} = (S^{\perp})^{\perp}$;

b) Deduce that

$$\overline{\operatorname{span}(S)} = \mathcal{H} \Leftrightarrow S^{\perp} = \{0\};$$

c) Prove that $\mathscr{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

Hint: You may use from Fourier Analysis that $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$.

Problem 11. Prove that if X is a normed space and Y a Banach space, then $\mathrm{B}(X, Y)$ is complete, i.e., a Banach space.

Problem 12. Let $\mathcal{H}$ be a Hilbert space, $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and $f \in \mathcal{H}$. Prove that the following are equivalent as $n \to \infty$:

(i) $f_n \to f$ in norm.

(ii) $f_n \rightharpoonup f$ weakly in $\mathcal{H}$ and $\|f_n\| \to \|f\|$ as real numbers.

Problem 13. Let $\mathcal{H}$ be a Hilbert space, $f \in \mathcal{H}$ and define a linear functional by

$$\Phi(f) : \mathcal{H} \to \mathbb{C}, \qquad g \mapsto \langle f, g\rangle.$$

Show that

$$\|\Phi(f)\| = \|f\|.$$

Problem 14. The adjoint has the following properties for $A, B \in \mathrm{B}(\mathcal{H})$ and $z \in \mathbb{C}$

a) $(A + zB)^* = A^* + \overline{z}B^*$;

b) $(AB)^* = B^*A^*$;

c) $(A^*)^* = A$

d) $\ker A^* = (\operatorname{ran} A)^\perp$ and $\ker A = (\operatorname{ran} A^*)^\perp$.

Problem 15 ((The Lax-Milgram Theorem))**.** Let $\mathcal{H}$ be a Hilbert space and

$$\alpha : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

a sesquilinear form. Asssume that

- α is *bounded*: there exists $C > 0$ so that for all $f, g \in \mathcal{H}$

$$|\alpha(f, g)| \leq C\|f\|\|g\|;$$

- α is *coercive*: there exists $a > 0$ so that for all $f \in \mathcal{H}$

$$\alpha(f, f) \geq a\|f\|^2.$$

Prove that:

a) There exists $A \in \mathrm{B}(\mathcal{H})$ so that $\alpha(f, g) = \langle Af, g\rangle$;

b) A is bijective with bounded inverse satisfying $\|A^{-1}\| \leq a^{-1}$;

c) $g = A^{-1}f$ is the unique minimiser of

$$g \mapsto \alpha(g, g) - 2\mathrm{Re}\langle f, g\rangle.$$

Problem 16. Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathrm{B}(\mathcal{H})$. Show that A is self-adjoint if and only if for all $f \in \mathcal{H}$

$$\langle f, Af\rangle \in \mathbb{R}.$$

Problem 17. Let, for $t > 0$, $T_t \in \mathrm{B}(L^2(\mathbb{R}))$ be the solution map of the heat equation on $L^2(\mathbb{R})$,

$$(T_t f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-|x-y|^2/(4t)} f(y)\mathrm{d}y.$$

Show that for all $f \in \mathcal{H}$, $t > 0$ we have $\|T_t f\| < \|f\|$, and $\|T_t\| = 1$.

Problem 18. Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ be bounded from below, i.e. $V(x) \geq -M$ for some $M \geq 0$ and a.e. $x \in \mathbb{R}^d$.

a) Prove that for every $f \in L^2(\mathbb{R}^d)$ and $\lambda > M$ there exists a unique $u \in H^1(\mathbb{R}^d)$ such that
$$\forall \varphi \in H^1(\mathbb{R}^d) : \langle \nabla u, \nabla \varphi \rangle + \langle (V + \lambda) u, \varphi \rangle = \langle f, \varphi \rangle,$$
that is, there is a unique weak solution to the equation
$$-\Delta u(x) + V(x)u(x) + \lambda u(x) = f(x).$$

b) Prove that the weak solution $u \in H^1(\mathbb{R}^d)$ obtained in part a) is an element of $H^2(\mathbb{R}^d)$.

Problem 19. Let $A \in \mathrm{B}(\mathcal{H})$ and show that $\sigma(A)$ is compact.

Problem 20. Let A, $D(A)$ be densely defined. Show that if $\rho(A) \neq \emptyset$, then A is closed.

Hint: Consider the set
$$\{(f, g) \in \mathcal{H} \times \mathcal{H} : (g, f) \in \mathscr{G}(A)\}.$$

Problem 21. For a (possibly unbounded) measurable function $g : \mathbb{R}^d \to \mathbb{C}$ consider the linear map M_g in $L^2(\mathbb{R}^d)$ defined by
$$\begin{aligned} \mathcal{D}(M_g) &:= \{f \in L^2(\mathbb{R}^d) \,|\, gf \in L^2(\mathbb{R}^d)\} \\ (M_g f)(x) &:= g(x) f(x)\,. \end{aligned}$$

Prove:

a) $\mathcal{D}(M_g)$ is dense in $L^2(\mathbb{R}^d)$.

b) $(M_g)^* = M_{\overline{g}}$.

c) M_g is closed.

d) If $g \in L^\infty(\mathbb{R}^d)$ then M_g is bounded, and
$$\|M_g\| = \|g\|_\infty = \sup\Big\{t : |\{x \in \mathbb{R}^d : |g(x)| \geq t\}| > 0\Big\},$$
where $|V|$ denotes the Lebesgue measure of a measurable subset $V \subset \mathbb{R}^d$.

e) M_g is not bounded if $g \notin L^\infty(\mathbb{R}^d)$.

Problem 22. Let $A \in \mathrm{B}(\mathbb{C}^d) = \mathbb{C}^{d \times d}$ and consider the linear autonomous ODE
$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t).$$

Show that
$$\limsup_{t \to \infty} |u(t)| < \infty$$
holds for all solutions if and only if all eigenvalues of A have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomial growth.

Problem 23 (The spectrum of a multiplication operator). Let $g : \mathbb{R} \to \mathbb{C}$ be measurable, M_g the operator of multiplication with g from Exercise 7.2 and denote by $|B|$ the Lebesgue measure of $B \in \mathscr{B}(\mathbb{R})$. Show that

a) $\sigma(M_g) = \operatorname{essran} g = \Big\{z \in \mathbb{C} \,|\, \forall \varepsilon > 0 : \Big|\{x \in \mathbb{R} \,|\, |z - g(x)| < \varepsilon\}\Big| > 0\Big\}$;

b) $z \in \mathbb{C}$ is an eigenvalue of M_g if and only if

$$|g^{-1}(\{z\})| = |\{x \in \mathbb{R} : g(x) = z\}| > 0;$$

c) Let $g(x) := x \ \ \forall x \in \mathbb{R}$. Then the quantum mechanical position operator $q := M_g$ is self-adjoint, has no eigenvalues, and $\sigma(q) = \mathbb{R}$.

Problem 24 (Dissipative matrices). Let $d \in \mathbb{N}$ and $A \in \mathrm{B}(\mathbb{C}^d)$ be a $d \times d$ matrix.

a) Assume there exists a unitary $U \in \mathrm{B}(\mathbb{C}^d)$ so that UAU^* is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for A to be dissipative.

b) Let $d = 2$ and A be the non-trivial Jordan block

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for A to be dissipative.

c) Let A be as in part *b)* and $\mathrm{Re}\lambda < 0$. Show that there exists a matrix S such that $B = SAS^{-1}$ is dissipative.

Problem 25. Let A be maximal dissipative and $\lambda > 0$. Prove that

$$\|AR_\lambda(A)\| \leq 1.$$

Problem 26 (The wave equation). In this exercise we solve the wave equation on $\mathbb{R}^d$ using the Hille Yosida theorem. The wave equation is

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ \qquad u(0) = u_0 \\ \quad \partial_t u(0) = \dot{u}_0. \end{cases} \tag{W}$$

a) Let $\mathcal{H} = H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let A be the operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

with domain $D(A) = H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \tag{A}$$

then u solves the wave equation (W).

b) Show that (u, v) solves (A) if and only if $(\tilde{u}, \tilde{v}) = \mathrm{e}^{-t}(u, v)$ solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$

c) Show that $A - 1$ is maximal dissipative.

d) State the existence and uniqueness result for the wave equation implied by a)–c) and the Hille-Yosida theorem, specifying the functional space for the solution u.

Problem 27. Let $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with $D(A) = H^1(\mathbb{R})$.

a) Show that A is maximal dissipative.

b) Show that for $u_0 \in L^2(\mathbb{R})$

$$(\mathrm{e}^{At} u_0)(x) = u_0(x + t).$$

c) Determine the spectrum of $T(t) = \mathrm{e}^{tA}$ and its decomposition into σ_p, σ_c, σ_r.

Problem 28. Let $a \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $a(x) \geq 1$ for all $x \in \mathbb{R}$ and $a, \frac{\mathrm{d}a}{\mathrm{d}x} \in L^\infty(\mathbb{R})$. Prove that for $u_0 \in H^2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \partial_t u(t, x) = \partial_x a(x) \partial_x u(t, x) + \partial_x u(t, x) \\ \qquad u(0) = u_0 \end{cases}$$

admits a unique solution

$$u \in C^1([0, \infty), L^2(\mathbb{R})) \cap C^0([0, \infty), H^2(\mathbb{R})).$$

Problem 29. We set for $f \in \mathscr{S}(\mathbb{R})$

$$\text{p.v.}\Big(\frac{1}{x}\Big)(f) := \lim_{\varepsilon \to 0} \int_{-\infty}^{\varepsilon} \frac{f(x)}{x} \mathrm{d}x + \int_{\varepsilon}^{\infty} \frac{f(x)}{x} \mathrm{d}x.$$

a) Show that $\text{p.v.}\big(\frac{1}{x}\big)$ is a well-defined tempered distribution.

b) Show that $\text{p.v.}\big(\frac{1}{x}\big)$ extends to a continuous linear functional on $H^2(\mathbb{R}^d)$.

Problem 30. Let $\mathcal{H}$ be a Hilbert space and A, $D(A)$ be a maximal dissipative operator on $\mathcal{H}$. Let $T(t) := \mathrm{e}^{tA}$, for $t \geq 0$, be the solution operator to the abstract Cauchy problem with generator A. Then $D(A)$ is characterised by

$$D(A) = \big\{ f \in \mathcal{H} : h^{-1}(\mathrm{e}^{hA} f - f) \text{ has a limit in } \mathcal{H} \text{ for } h \to 0 \big\}.$$

a) Prove that for $z \in \mathbb{C}$

$$\ker(A - z) = \bigcap_{t \geq 0} \ker(\mathrm{e}^{tA} - \mathrm{e}^{tz}).$$

b) Give an example of an operator A with $\ker(A) = \{0\}$, for which there exists a time $t > 0$ such that $\ker(\mathrm{e}^{tA} - 1) \neq \{0\}$.

B. Notation

Symbol	Explanation	Page
$\mathbb{N}$	Natural numbers (not including zero!)	
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$	
D	Differential of a vector-valued function	
grad	Gradient of a scalar function, $\operatorname{grad} f = Df$	
div	Divergence of a vector field, $\operatorname{div} v = \operatorname{Tr}(Dv)$	
$B(x,r)$	Open ball of radius r around x	15
$\mathscr{S}(\mathbb{R}^d)$	Space of Schwartz functions on $\mathbb{R}^d$	5
$\mathscr{S}'(\mathbb{R}^d)$	Space of tempered distributions on $\mathbb{R}^d$	7
$H^k(\mathbb{R}^d)$	Sobolev space of functions in $L^2(\mathbb{R}^d)$ with k weak derivatives in L^2	10
X	Usually a complex Banach space	16
$\mathrm{B}(X,Y)$	Banach space of bounded linear operators from X to Y	22
$\mathrm{B}(X)$	Banach space of bounded linear operators from X to X	22
X'	Space of continuous linear functionals on X (=$\mathrm{B}(X,\mathbb{C})$)	22
$\mathcal{H}$	Complex (separable) Hilbert space	17
$A, D(A)$	Densely defined linear operator	28
$\mathscr{G}(A)$	Graph of A	28
$\overline{A}$	Closure of $(A, D(A))$	28
$\lVert\cdot\rVert_{D(A)}$	Graph norm on $D(A)$	29
A^*	(Hilbert-) adjoint of $(A, D(A))$	23,30
$\ker(A)$	Kernel of A	
$\operatorname{ran}(A)$	Range of A	
$\rho(A)$	Resolvent set of A	33
$R_z(A)$	Resolvent of A in $z \in \rho(A)$, $(A-z)^{-1}$	33
$\sigma(A)$	Spectrum of A	33
$C^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$	
$C_0^k(U)$	Space of k-times continuously differentiable functions $U \to \mathbb{C}$ with compact support, $\operatorname{supp} f \Subset U$	

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