

Sheet 11 Exam preparation

You will have **180 minutes** in total to complete the examination. The following rules will apply:

- Switch off your mobile phone and put it away. Do not leave it on the table.
- You may use the lecture notes of the course in paper form that may include your own annotations. No other material is admitted.
- Do not write with a pencil, nor use the colours red or green.
- All answers and solutions must provide sufficiently detailed arguments.
- Only one solution to each problem will be accepted. Please cross out everything which is not supposed to count.

Problem 11.1.

[5 Points]

Mark all correct statements.

a) Let $A, D(A)$ be the densely defined operator

$$Af := -\frac{d^2 f}{dx^2} - i\frac{df}{dx}$$

with domain $D(A) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}^d) =: \mathcal{H}$. Then A is

- dissipative;
- self-adjoint;

b) The solution operator to the heat equation on \mathbb{R}^d , $T(t) := e^{t\Delta} \in \mathcal{B}(L^2(\mathbb{R}^d))$, $t \geq 0$,

- is closed;
- is unitary;
- has spectrum $\sigma(T(t)) = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 11.2.

[5 Points]

Let $a \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $a(x) \geq 1$ for all $x \in \mathbb{R}$ and $a, \frac{da}{dx} \in L^\infty(\mathbb{R})$. Prove that for $u_0 \in H^2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \partial_t u(t, x) = \partial_x a(x) \partial_x u(t, x) + \partial_x u(t, x) \\ u(0) = u_0 \end{cases}$$

admits a unique solution

$$u \in C^1([0, \infty), L^2(\mathbb{R})) \cap C^0([0, \infty), H^2(\mathbb{R})).$$

Problem 11.3.

[5 Points]

We set for $f \in \mathcal{S}(\mathbb{R})$

$$\text{p.v.} \left(\frac{1}{x} \right) (f) := \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x} dx.$$

a) Show that $\text{p.v.} \left(\frac{1}{x} \right)$ is a well-defined tempered distribution. [3 Points]

b) Show that $\text{p.v.} \left(\frac{1}{x} \right)$ extends to a continuous linear functional on $H^2(\mathbb{R})$.

[2 Points]

Problem 11.4.

[5 Points]

Let \mathcal{H} be a Hilbert space and $A, D(A)$ be a maximal dissipative operator on \mathcal{H} . Let $T(t) := e^{tA}$, for $t \geq 0$, be the solution operator to the abstract Cauchy problem with generator A . Then $D(A)$ is characterised by

$$D(A) = \{f \in \mathcal{H} : h^{-1}(e^{hA}f - f) \text{ has a limit in } \mathcal{H} \text{ for } h \rightarrow 0\}.$$

a) Prove that for $z \in \mathbb{C}$ [4 Points]

$$\ker(A - z) = \bigcap_{t \geq 0} \ker(e^{tA} - e^{tz}).$$

b) Give an example of an operator A with $\ker(A) = \{0\}$, for which there exists a time $t > 0$ such that $\ker(e^{tA} - 1) \neq \{0\}$. [1 Points]