Sheet 3

Exercise 3.1

Let $k \in \mathbb{N}_0$, $f \in H^k(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d)$. Prove that $fg \in H^k(\mathbb{R}^d)$ and the generalised Leibniz rule holds for the derivatives of order $|\alpha| \le k$.

Exercise 3.2

Let X be a normed space and $S \subset X$. A point $x \in X$ is called a limit point of S if there exists a sequence in $x_n \in S$, $n \in \mathbb{N}$ that converges to x. Prove that

a) S is closed if and only if it contains all its limit points;

b) The closure $\overline{S}$ is the union of S and its limit points;

c) Assume that X is complete and $Y \subset X$ a subspace. Then Y is complete if and only if Y is closed.

Homework

Exercise 3.3

a) Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}\mathrm{e}^{-|x|}$. Show that g (more precisely the associated distribution, φ_g) is the unique solution in $\mathscr{S}'(\mathbb{R})$ to the equation

$$(1 - \Delta)\varphi = \delta_0\,,$$

by

1) the Fourier transform;
2) directly using the distributional derivative.

b) Prove that for $f \in \mathscr{S}(\mathbb{R})$ the unique solution to the equation

$$(1 - \Delta)u = f$$

is

$$u(x) = \int g(x - y)f(y)\mathrm{d}y.$$

Remark: g is called the fundamental solution or Green's function for the equation.

Hand in on 08.02.2022 before the lecture or by 10:00 by mail to `jonas.lampart@u-bourgogne.fr`.