Sheet 2

Exercise 2.1 (Multiplication and convolution on $\mathscr{S}'$)

Let $g \in \mathscr{S}(\mathbb{R}^d)$ and define the multiplication by g as a map $(M_g f)(x) := g(x)f(x)$.

a) Show that $M_g' : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is linear and continuous;

b) For $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ define multiplication with g by $g\varphi := M_{\overline{g}}'\varphi$ and show that

$$\mathscr{F}(g\varphi) = (2\pi)^{-d/2}\hat{g} * \hat{\varphi},$$

where $*$ is the convolution of $\hat{g} \in \mathscr{S}(\mathbb{R}^d)$ with $\hat{\varphi} \in \mathscr{S}'(\mathbb{R}^d)$ defined in the lecture.

Exercise 2.2 (The Fourier transform of complex Gaussians II)

Let $t > 0$ and set for $\varepsilon \geq 0$

$$f_\varepsilon(x) := \mathrm{e}^{-(\mathrm{i}t+\varepsilon)x^2/2}.$$

a) Show that $f_\varepsilon \to f_0$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \to 0$;

b) Show that

$$\hat{f}_0(p) = \frac{\mathrm{e}^{\mathrm{i}\frac{p^2}{2t}}}{\sqrt{\mathrm{i}t}}.$$

Can you explain the relation to Exercise 1.4?

Hint: We know from Exercise 1.2 that for $\varepsilon > 0$

$$\hat{f}_\varepsilon(p) = \frac{\mathrm{e}^{-\frac{p^2}{2(\mathrm{i}t+\varepsilon)}}}{\sqrt{\mathrm{i}t+\varepsilon}}.$$

Homework

Exercise 2.3 (The delta distribution)

Define for $f \in \mathscr{S}(\mathbb{R}^d)$

$$\delta_0(f) := f(0).$$

Let $g \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g = 1$ and define, for $\epsilon > 0$, $g_\epsilon(x) := \epsilon^{-d} g(\epsilon^{-1}x)$. Prove:

a) For every $\epsilon > 0$, g_ϵ defines a tempered distribution $\varphi_\epsilon \in \mathscr{S}'(\mathbb{R}^d)$ via

$$\varphi_\epsilon(f) := \int_{\mathbb{R}^d} \overline{g_\epsilon(x)} f(x) \mathrm{d}x, \qquad f \in \mathscr{S}(\mathbb{R}^d).$$

That is, $f \mapsto \varphi_\varepsilon(f)$ is linear and continuous on $\mathscr{S}(\mathbb{R}^d)$;

b) As $\epsilon \to 0$, $\varphi_\epsilon \to \delta_0$ in $\mathscr{S}'(\mathbb{R}^d)$;

c) Let $\theta \in \mathscr{S}'(\mathbb{R})$ be defined by $\theta(f) := \int_{\mathbb{R}} 1_{[0,\infty)}(x) f(x) \mathrm{d}x$. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\theta = \delta_0$. (Here $\frac{\mathrm{d}}{\mathrm{d}x} := (\partial^1)_{\mathscr{S}'}$ is the distributional derivative as defined in the lecture).

Hand in on 01.02.2023 before the lecture or by 10:00 by mail to `jonas.lampart@u-bourgogne.fr`.