Sheet 1

Exercise 1.1

Use the Fourier transform to find a solution to the equation

$$\begin{cases} \partial_t u(t,x) - \partial_x u(t,x) = 0 \\ u(0,x) = u_0(x). \end{cases}$$

with $u_0 \in \mathscr{S}(\mathbb{R})$.

Exercise 1.2 (The Fourier transform of complex Gaussians)

a) Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. Show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}}.$$

b) Calculate the Fourier transform of $f(x) = \mathrm{e}^{-ax^2}$ for $\mathrm{Re}(a) > 0$.
Hint: Use Cauchy's theorem from complex analysis.

Exercise 1.3 (The free Schrödinger equation)

In this exercise we show that the free Schrödinger equation in one dimension

$$\mathrm{i}\partial_t u(t,x) = -\tfrac{1}{2}\partial_x^2 u(t,x)$$

is solved for $t > 0$ by

$$u(t,x) = \frac{1}{\sqrt{2\pi \mathrm{i} t}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{(x-y)^2}{2t}} u_0(y) \mathrm{d}y,$$

for any $u_0 \in \mathscr{S}(\mathbb{R})$.

a) Show that for $(t,x) \in (0,\infty) \times \mathbb{R}$, $u(t,x)$ is continuously differentiable in t and twice continuously differentiable in x.

b) Show that $u(t,x)$ solves the Schrödinger equation for $(t,x) \in (0,\infty) \times \mathbb{R}$.

Homework: Hand in on 25.01.2022 before the lecture or by 10:00 by mail to `jonas.lampart@u-bourgogne.fr`.